Deriving the equivalent circuit of a Tesla magnifier

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https://bartmcguyer.com/notes/note-17-MagnifierEquations.pdf

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TL;DR: A toy-model derivation of an equivalent circuit for a three-circuit version of a Tesla coil nicknamed a Tesla magnifier. Approximates both the secondary and tertiary coils as uniform transmission lines, recovers previous results for the secondary, and predicts three self-capacitances for the tertiary, one of which is negative.

Continuing after Refs. 1 and 2, which derived an equivalent circuit for a traditional two-circuit Tesla transformer (or Tesla coil), this Note derives a lumped-element equivalent circuit for a three-circuit variation of a Tesla transformer often called a Tesla magnifier.[3] (For more about magnifiers, please see Ref. 4.) Fig. 1(a) shows the physical setup of a magnifier with primary, secondary, and tertiary coils. To proceed, this Note approximates[4] the secondary and tertiary coils as uniform transmission lines and their electrodes and inter connections as lumped capacitances, as shown in Fig. 1(b), which leads to the equivalent circuit in Fig. 1(c).

What's interesting are the "self" capacitances that appear as corrections in what otherwise closely resembles a low-frequency (or dc) equivalent circuit. While much attention's been paid to the self capacitance of the secondary,[5] comparatively little has been paid to those of the tertiary. This Note derives three self capacitances for the tertiary, labeled $C_{t,1}$, $C_{t,2}$, and $C_{t,3}$ in Fig. 1(c), and, surprisingly, suggests that $C_{t,3}$ should be negative (agreeing with Ref. 6). This Note also provides support for the results of Ref. 2.

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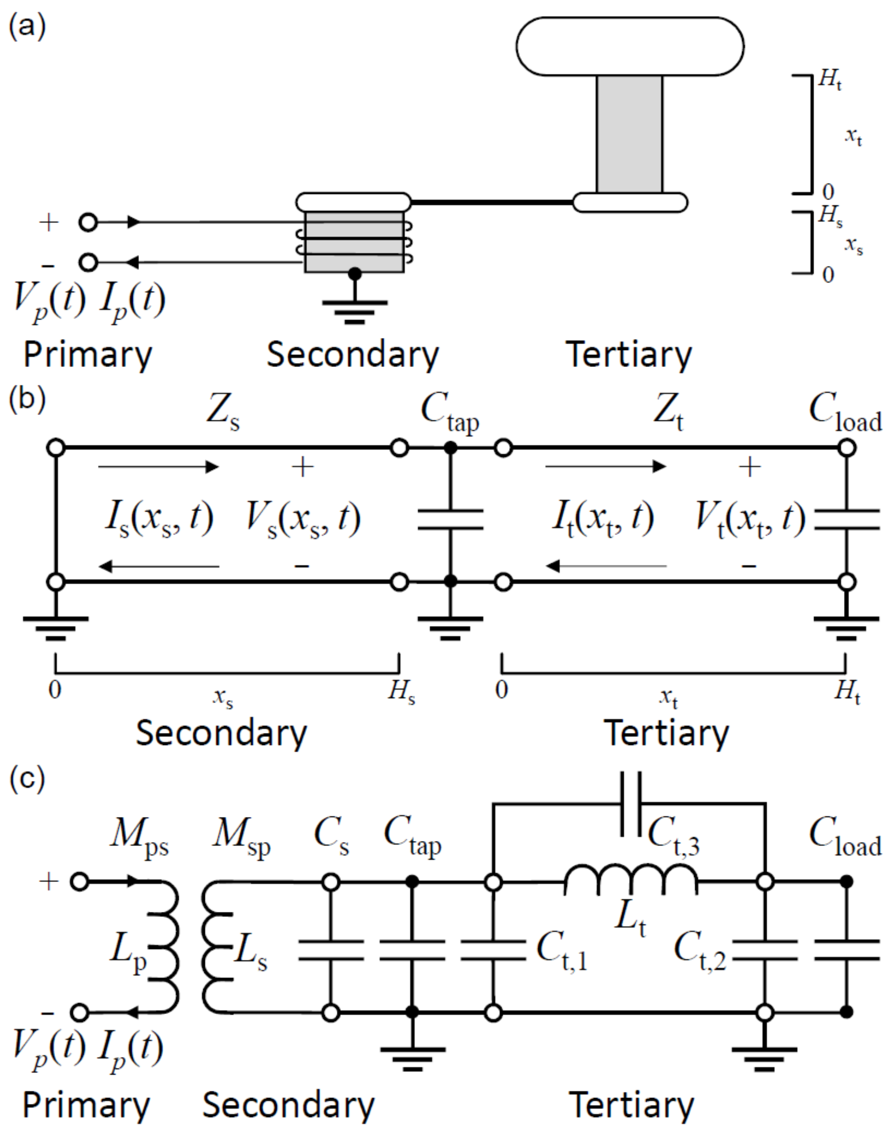

FIG. 1. Tesla magnifier setup, transmission-line modeling, and equivalent-circuit results. (a) Physical setup showing side profiles of the primary coil, the secondary coil with its top electrode, the secondary-tertiary connection, and the tertiary coil with its top and bottom electrodes. (b) Model approximating the secondary and tertiary coils each as uniform transmission lines, and their electrodes and inter connection as two lumped capacitances, C_{tap} and C_{load}, following Ref. 4. (c) Equivalent circuit result with parameter values given in the text below and losses (resistors) ignored for simplicity. Surprisingly, the derivation predicts that $C_{t,3}$ is negative.

I. DERIVATION

A. Tertiary equivalent circuit

Let's begin with the most interesting part, the single-layer solenoid used for the tertiary coil. Note that if we use that particular solenoid as a secondary, meaning with a grounded base, and we assume a nearly uniform current profile, then we could use the traditional equivalent circuit for a solenoid (or the so-called classical model of an inductor). However, when we use it as a tertiary, then its base is no longer grounded, which violates an important assumption of that equivalent circuit. Therefore, we need to derive a new equivalent circuit.

1. Stored-energy approach

Let's first extend an energy argument from previous work on the secondary.[2,7,8] Consider a tertiary solenoid of height H_t with a linear voltage profile,

$$V(x,t) = [V_1 + (V_2 - V_1)x/H_t]\cos(\omega t), \tag{1}$$

where $x \in [0, H_t]$, $V(0,t) = V_1(t) = V_1\cos(\omega t)$ is the base voltage, and $V(H_t,t) = V_2(t) = V_2\cos(\omega t)$ is the top voltage. Assuming a uniform distributed shunt capacitance c_t along the tertiary solenoid, and using $V_1(V_2 - V_1) = [V_2^2 - V_1^2 - (V_2 - V_1)^2]/2$, the energy stored by this profile is then

$$U = \frac{c_t}{2}\int_0^{H_t} \left\langle V(x,t)^2\right\rangle dx = \frac{c_t H_t}{4}\left(\left\langle V_1(t)^2\right\rangle + \left\langle V_2(t)^2\right\rangle - \frac{1}{3}\left\langle [V_2(t) - V_1(t)]^2\right\rangle\right) \tag{2}$$

$$= \frac{C_{t,1}}{2}\left\langle V_1(t)^2\right\rangle + \frac{C_{t,2}}{2}\left\langle V_2(t)^2\right\rangle + \frac{C_{t,3}}{2}\left\langle [V_2(t) - V_1(t)]^2\right\rangle, \tag{3}$$

where brackets denote a time average and the new capacitances are

$$C_{t,1} = \frac{1}{2}C_{t,0}, \tag{4}$$

$$C_{t,2} = C_{t,1}, \text{ and} \tag{5}$$

$$C_{t,3} = -\frac{1}{6}C_{t,0}, \text{ using} \tag{6}$$

$$C_{t,0} = c_t H_t. \tag{7}$$

As written, the energy has three terms that correspond to the Delta (or Pi) circuit in Fig. 1(c): (i) a shunt capacitance of $C_{t,1}$ from the base to ground, (ii) a shunt capacitance of $C_{t,2}$ from the top to ground, and (iii) a series (axial, longitudinal, or mutual) capacitance of $C_{t,3}$ from the top to the base.

What's particularly surprising is that the mutual capacitance $C_{t,3}$ is negative (!). However, this makes sense considering how energy's stored by a linear voltage profile. That is, if you neglect this capacitance (set $C_{t,3} \to 0$), then U overestimates the energy stored by the profile. To correct for this, an amount has to be removed, which requires a negative capacitance. (This negativity does not seem to be related to the negative coefficients of induction in the electrostatic capacitance matrix.[7])

As a consistency check, note that if the base is grounded ($V_1 = 0$), then this reproduces the Miller self-capacitance for the secondary, which is a shunt capacitance of $C_{t,2} + C_{t,3} = C_{t,0}/3$ from top to ground.[9] Additionally, a more careful derivation allowing a nonuniform distributed capacitance $c_t(x, y)$ following the approach of Ref. 7 (see p. 9) gives the same results as above, though it also gives a correction term that's the same as that for a secondary given in Eq. (41) of that reference.

Before we continue, it's interesting to consider the dual case of a linear current profile,

$$I(x, t) = [I_1 + (I_2 - I_1)x/H_t] \cos(\omega t), \tag{8}$$

and a uniform distributed series inductance l_t. The energy stored by this profile follows in the same way, giving

$$U = \frac{l_t}{2} \int_0^{H_t} \left\langle I(x,t)^2 \right\rangle dx = \frac{l_t H_t}{4} \left(\left\langle I_1(t)^2 \right\rangle + \left\langle I_2(t)^2 \right\rangle - \frac{1}{3} \left\langle [I_2(t) - I_1(t)]^2 \right\rangle \right) \tag{9}$$

$$= \frac{L_{t,1}}{2} \left\langle I_1(t)^2 \right\rangle + \frac{L_{t,2}}{2} \left\langle V_2(t)^2 \right\rangle + \frac{L_{t,3}}{2} \left\langle [I_2(t) - I_1(t)]^2 \right\rangle, \tag{10}$$

where the new inductances are

$$L_{t,1} = \frac{1}{2} L_t, \tag{11}$$

$$L_{t,2} = L_{t,1}, \text{ and} \tag{12}$$

$$L_{t,3} = -\frac{1}{6} L_t, \text{ using} \tag{13}$$

$$L_t = L_{t,0} = l_t H_t. \tag{14}$$

In this case, the energy corresponds to a Wye (or Tee) circuit with two series inductances $L_{t,1}$ and $L_{t,2}$ and a shunt inductance $L_{t,3}$ connecting their midpoint to ground. If one end is open circuit (e.g., $I_2 = 0$), then this reproduces the Miller self-inductance for an antenna,[9] which is a series inductance of $L_{t,1} + L_{t,3} = L_{t,0}/3$. Again, surprisingly, one of these inductances is negative.

However, since we assume nearly spatially uniform current in the tertiary, let's save the dual case of linearly varying current for future work to explore.

2. Low-frequency, non-resonant approach

We can approximately derive the linear-voltage results of the last section using an $ABCD$ solution[10,11] for a transmission line, which is provided in Apendix A, as follows. For similar derivations, see Refs. 6 and 11.

Consider a Delta circuit with three nodes indexed by n, each with voltage V_n. Let $n = 1$ and $n = 2$ correspond to the bottom and top of the tertiary solenoid, respectively, and $n = 3$ to ground ($V_3 = 0$). Let impedances Z_{nm} connect each pair of nodes n and m and let currents I_{nm} flow from node n to node m. The Kirchoff circuit equations for this Delta circuit are

$$V_2 - V_1 = -Z_{12} I_{12} \tag{15}$$
$$V_1 = Z_{13} I_{13} \tag{16}$$
$$V_2 = Z_{23} I_{23}. \tag{17}$$

Anticipating last section's result, let $Z_{13} = Z_{23}$.

Appendix A provides an $ABCD$ solution with the form

$$\begin{pmatrix} V(H_t, t;\omega) \\ I(H_t, t;\omega) \end{pmatrix} = \begin{pmatrix} A(H_t;\omega) & B(H_t;\omega) \\ C(H_t;\omega) & D(H_t;\omega) \end{pmatrix} \begin{pmatrix} V(0, t;\omega) \\ I(0, t;\omega) \end{pmatrix}. \tag{18}$$

Connecting the circuit variables with the line gives a corresponding $ABCD$ form as

$$\begin{pmatrix} V_2 \\ I_{12} - I_{23} \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} V_1 \\ -I_{12} - I_{13} \end{pmatrix}, \tag{19}$$

where the $ABCD$ parameters for the Delta circuit are

$$A' = D' = 1 + Z_{12}/Z_{13} \tag{20}$$
$$B' = -Z_{12} \tag{21}$$
$$C' = \left(\frac{-1}{Z_{13}}\right)\left(2 + \frac{Z_{12}}{Z_{13}}\right). \tag{22}$$

This follows from computing $I_{12} + I_{13}$ to get A' and B', and then computing $I_{12} - I_{23}$ to get C' and D' after substituting for V_2 to insert A' and B'.

These two $ABCD$ forms are equivalent if their parameters are identical. Equating $B' = B(H_t;\omega)$ gives

$$Z_{12} = Z_0 \sinh(\gamma H_t). \tag{23}$$

Equating $A' = A(H_t;\omega)$, or equivalently $D' = D(H_t;\omega)$, gives

$$Z_{13} = \frac{Z_{12}}{\cosh(\gamma H_t) - 1} = Z_0 \coth(\gamma H_t/2). \tag{24}$$

Last, $C' = C(H_t;\omega)$ follows from the identity $\cosh(x)^2 - \sinh(x)^2 = 1$.

Finally, if we assume a low spatial frequency, such that γH_t is small, then we can use $\sinh(x) \approx 1/(1/x - x/6)$, $\coth(x) \approx 1/x + x/3$, $Z_0\gamma = z = r + i\omega l_t$, and $\gamma/Z_0 = y = i\omega c_t$ to simplify these results, giving:

$$Z_{13} \approx \frac{2Z_0}{\gamma H_t} + \frac{\gamma H_t Z_0}{6} = \frac{1}{i\omega(C_{t,0}/2)} + \frac{R_t + i\omega L_t}{6} \approx \frac{1}{i\omega(C_{t,0}/2)} \tag{25}$$
$$Z_{12} \approx \frac{Z_0}{\frac{1}{\gamma H_t} - \frac{\gamma H_t}{6}} = \frac{1}{\frac{1}{R_t + i\omega L_t} + i\omega(\frac{-C_{t,0}}{6})}, \tag{26}$$

where $C_{t,0} = c_t H_t$ and $L_t = l_t H_t$ as introduced earlier. The top line recovers $C_{t,1} = C_{t,2} = C_{t,0}/2$, and the bottom line $C_{t,3} = -C_{t,0}/6$, giving the equivalent circuit in Fig. 1(c). Note that each line was expanded only through the first capacitive correction, which was the first term for the top line and the second term for the bottom line. Ref. 6 provides higher-order terms, and also derives the same negative value of $C_{t,3}$.

B. Secondary equivalent circuit, revisited

My previous note, Ref. 2, only considered a secondary with a capacitive load. The secondary in a magnifier has a more complicated load, so it's not certain the same results hold. Let's revisit the secondary to show that they should, as you'd expect.

1. Grounded-tertiary approach

First, notice that we didn't need to consider the complexity of the bottom load of the tertiary in the above derivation. So, assuming we have a correct equivalent circuit for the tertiary, we should be able to ground one of its ends to create a secondary equivalent circuit. As foreshadowed above, this recovers the Miller self-capacitance for the secondary found in previous work,[2] since $C_{t,1} + C_{t,3} = C_{t,0}/3$. Or, in other words, the tertiary equivalent circuit is a more general form of the secondary equivalent circuit.

2. Low-frequency sum-of-modes approach

A more straightforward approach is to derive an equivalent circuit for the secondary that does not make any particular assumption about its load. One way to do this is to use the approach of Ref. 1, specifically, starting with the top-direct-port form shown in its Fig. 6(b). Assuming we're only interested in frequencies lower than the fundamental self-resonance of an open-ended secondary, we can simplify and sum all of the mode equivalent circuits. (This approach cannot be new, but I have not found it elsewhere yet.)

In this case, the mode equivalent circuit are given by $\eta_\nu = \alpha_n u = 1/\sqrt{\phi_\nu}$, where $\phi_\nu = k_\nu H/2$ and $k_\nu = (2\nu - 1)\pi/(2H)$, and $\chi_\nu = \eta_n u^2$. This gives $A_\nu = \chi_\nu/k_\nu = 2/(k_\nu^2 H)$ and $B_\nu = 1/(\chi_\nu k_\nu) = H/2$, which in turn gives $L_\nu = A_\nu l = (lH)2/(k_\nu H)^2$ and $C_\nu = B_\nu c = (cH)/2$. (Resistance and mutual inductance with the primary coil follow the treatment of L_ν, but let's ignore those for simplicity.)

Next, we need to combine all of the mode circuits in series. The impedance of each is $Z_\nu = 1/[1/(i\omega L_\nu) + i\omega C_\nu] = i\omega L_n u/[1 - (\omega/\omega_\nu)^2]$, where $\omega_\nu = 1/\sqrt{L_\nu C_\nu}$. Assuming low frequencies, $Z_\nu \approx i\omega L_\nu(1 + \omega^2 L_\nu C_\nu)$, Or, put differently, $Z_\nu = 1/[1/Z_{L_\nu} + 1/Z_{C_\nu}] \approx Z_{L_\nu}(1 - Z_{L_\nu}/Z_{C_\nu})$. Therefore, the total series impedance across all modes is $\sum_{\nu=1}^{\infty} Z_\nu \approx \sum_{\nu=1}^{\infty} Z_{L_\nu} - \sum_{\nu=1}^{\infty} Z_{L_\nu}^2/Z_{C_\nu} = Z_{L_\Sigma}(1 - Z_{L_\Sigma}/Z_{C_\Sigma}) \approx 1/[1/Z_{L_\Sigma} + 1/Z_{C_\Sigma}]$, where the last step reversed the previous approximation. This gives an equivalent inductance of $L_\Sigma = \sum_{\nu=1}^{\infty} L_\nu = lH = L_s$, using the comment after Eq. (26) in that reference. (Resistance and mutual inductance would follow similarly.) Finally, the equivalent capacitance is $C_\Sigma = \sum_{\nu=1}^{\infty} C_\nu L_\nu^2/(L_d c)^2 = cH/3$, the Miller result, because $\sum_{\nu=1}^{\infty}(k_\nu \phi_\nu)^{-2} = 2H/3$.

C. Full equivalent circuit for secondary and tertiary

At this point, we've derived equivalent circuits for the secondary and for the tertiary, separately. Originally, I'd hoped to derive an equivalent circuit for the combined secondary with tertiary circuit, in case it'd be simpler to derive, or perhaps more accurate. However, I tried many different approaches based on Refs. 1 and 2, and all encountered difficulties.

Changing directions, let's instead prepare to test the results so far.

1. Resonant frequencies

Let's derive the resonant frequencies of the equivalent circuit for the secondary and tertiary only (ignoring the primary) in Fig. 1(c), so we can numerically test them in the next section. Introducing the shorthand $C_a = C_s + C_{t,1} + C_{\text{tap}}$, $C_b = C_{t,2} + C_{\text{load}}$, and $C_c = C_{t,3}$, then the equivalent circuit gives two equations: Two from voltage across inductors,

$$V_s = -L_s \frac{dI_s}{dt} \tag{27}$$

$$V_t - V_s = -L_t \frac{dI_s}{dt}, \tag{28}$$

and two from current conservation (left and right sides of L_t),

$$I_s - I_t - C_a \frac{dV_s}{dt} + C_c \left(\frac{dV_t}{dt} - \frac{dV_s}{dt} \right) = 0 \tag{29}$$

$$I_t - C_b \frac{dV_t}{dt} - C_c \left(\frac{dV_t}{dt} - \frac{dV_s}{dt} \right) = 0. \tag{30}$$

Since we're interested in resonance, we can assume all voltages and currents are proportional to $e^{i\omega t}$. Thus we can substitute the first two equations in to the last two to remove the voltages. Converting both to give ratios of current, and then setting them equal, gives an equation for the resonant frequencies:

$$\frac{I_s}{I_t} = \frac{1 - \omega^2 C_c L_t}{1 - \omega^2 C_a L_s} = \frac{1 - \omega^2 (C_b + C_c) L_t}{\omega^2 C_b L_s}. \tag{31}$$

Solving this (via Mathematica) gives two resonant frequencies:

$$\omega_\pm(C_a, C_b, C_c, L_s, L_t) = \sqrt{\frac{2}{\Delta \pm \sqrt{\Delta^2 - 4 L_s L_t (C_a C_b + C_a C_c + C_b C_c)}}}, \tag{32}$$

$$\text{where} \quad \Delta = (C_a + C_b) L_s + (C_b + C_c) L_t. \tag{33}$$

II. NUMERICAL TEST

To test the equivalent circuit of Fig. 1(c), let's compare the resonant frequencies it gives for the combined secondary and tertiary circuit in the last section with numerical and experimental results explored in Ref. 4.

First, let's test the identical line case from Ref. 4. Fig. 2 shows that the two resonant frequencies from the last section accurately reproduce the lowest-two numerical curves in

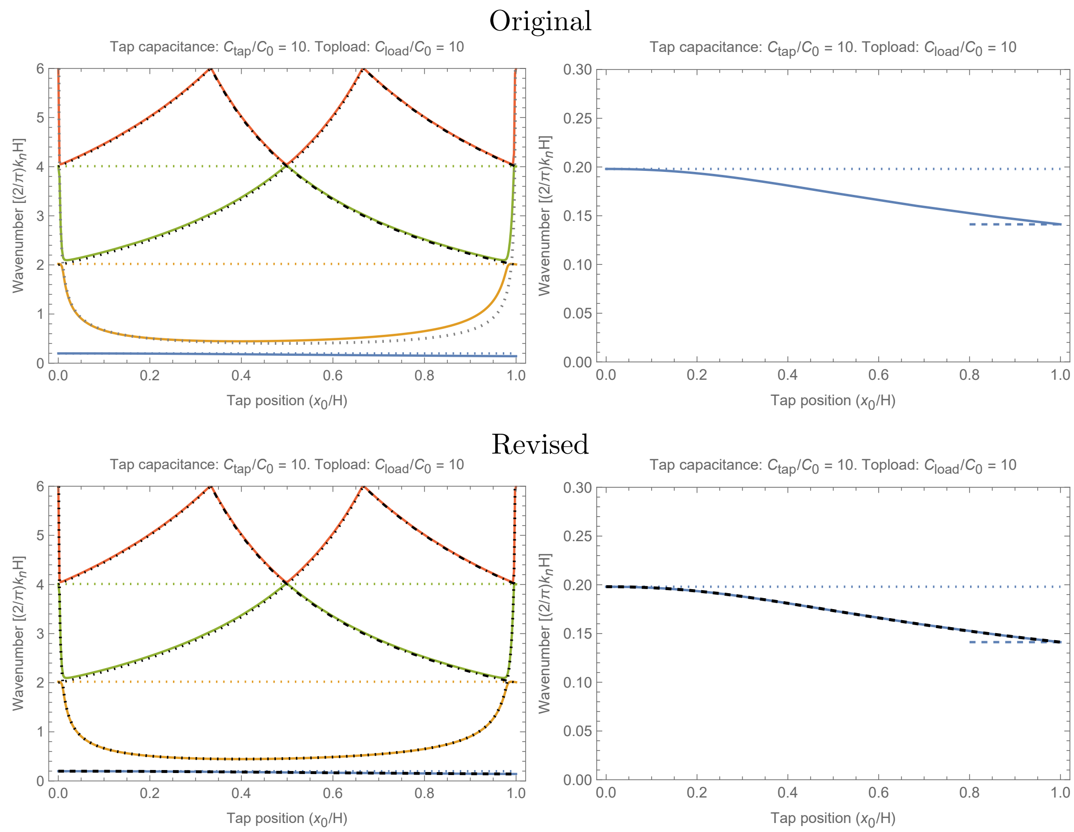

FIG. 2. Numerical test. (Top row) Reproduction of Fig. 4 from Ref. 4. (Top left) Solid curves are numerical solutions to the first few wavenumbers k_n for the case of identical lines with a topload. Notice that the grey dotted curve doesn't quite match on the bottom right side. (Top right) Highlight of k_1, which transitions between loaded quarter-wave modes with $C_{\text{tap}} = 0$ on the left and with $C_{\text{tap}} \neq 0$ on the right, shown with a partial-width dashed line. (Bottom row) Revised plots that include $k_\pm$. (Bottom left) The grey dotted curve was replaced with a black dotted curve corresponding to k_-, which matches well. (Bottom right) A new black dashed curve corresponding to k_+ agrees well.

that work's Fig. 4. Additionally, the upper resonance does a better job than an approximate formula given in that paper. In detail, letting $r = x_0/H$, $C_0 = cH$, and $L_0 = lH$, then this comparison comes from setting $C_s = rH/3$, $C_{t,1} = C_{t,2} = C_0(1-r)/2$, $C_{t,3} = -C_0(1-r)/6$, $L_s = rL_0$, and $L_t = (1-r)L_0$. This gives $C_a = C_{\text{tap}} + C_0(3-r)/6$, $C_b = C_{\text{load}} + C_0(1-r)/2$, and $C_c = C_{t,3} = -C_0(1-r)/6$, and finally $k_\pm H = \sqrt{L_0 C_0}\,\omega_\pm(C_a, C_b, C_c, L_s, L_t)$, which is a function of r, C_{tap}/C_0, and C_{load}/C_0, as plotted in Fig. 2.

Ref. 4 also compares its numerical results and available experimental results for the lowest few resonant frequencies of a particular magnifier. Table I reproduces Table I in that work, and adds four rows for this work. The first new row gives the "full" model, which uses the traditional estimate of $C_s = C^{(\text{secondary})}_{\text{Medhurst}}$ and then, following the Miller self-capacitance form, estimates the tertiary self capacitances using $C_{t,0} = 3C^{(\text{tertiary})}_{\text{Medhurst}}$, which gives $C_{t,1} = C_{t,2} = 1.5C^{(\text{tertiary})}_{\text{Medhurst}}$ and $C_{t,3} = -0.5C^{(\text{tertiary})}_{\text{Medhurst}}$. Otherwise, all parameters follow from

TABLE I. Numerical test. Modified version of Table I from Ref. 4 that compares different calculations of the lowest resonant frequencies of a 3:4:5 magnifier. The first three rows are unchanged, but the final four rows are new additions that follow from $k_{\pm}$, as explained in the text.

Source	f_a (kHz)	f_b (kHz)	f_c (kHz)	f_d (kHz)	System
Experiment (Ref. 12):	232	307	385	–	Primary, secondary, and tertiary.
Simulation (Ref. 13):	227	303	383	873	Primary, secondary, and tertiary.
Numerical (Ref. 4):	227	–	361	922	Secondary and tertiary only.
This work:	220	–	350	–	Full model.
This work:	213	–	319	–	No longitudinal self-cap ($C_{t,3} = 0$).
This work:	230	–	334	–	$C_{t,3} = 0$ & adjusted $C_{t,1}, C_{t,2}$.
This work:	278	–	415	–	No self-cap: $C_s = C_{t,0} = 0$.

that reference. The agreement with Ref. 4's row is rather decent, though it's hard to infer any significance beyond that, since there's so much uncertainty in the input parameters.

To explore more, the next three rows show the effect of modifying the "full" model. The second new row shows the results if $C_{t,3}$ is arbitrarily set to zero, which is poor. The third new row keeps that zero, but also adjusts the shunt capacitances to both act like secondary self capacitances, $C_{t,1} = C_{t,2} = C^{(\text{tertiary})}_{\text{Medhurst}}$, which helps a little. Finally, the third new row shows the effect of removing all self capacitances, for the secondary and tertiary, which is very poor.

III. DISCUSSION

I wrote this Note because a straightforward extension of the stored-energy approach that gives the Miller self-capacitance model for a secondary surprisingly predicted a negative capacitance for a self-capacitance across a tertiary. This curious negativity persisted despite attempts to remove it by transforming (see Appendix B). Investigating, it makes sense from energy conservation, and at least one prior work in the literature supports it.[6]

What remains is to see whether it's useful. The modeling here is very crude, so while I suspect the general prediction of three self capacitances is correct, I doubt their predicted values are accurate. For example, prior work has successfully used three such capacitances, though in a slightly different way, all with positive values.[14] The numerical tests presented here are encouraging, so if you want to give these results a try, you could estimate values for your case by appealing to the Medhurst empirical formula and the Miller self-capacitance model. This gives $C_{t,0} \approx 3C^{(\text{tertiary})}_{\text{Medhurst}}$, so estimates $C_{t,1} \approx C_{t,2} \approx 1.5C^{(\text{tertiary})}_{\text{Medhurst}}$ and $C_{t,3} \approx -0.5C^{(\text{tertiary})}_{\text{Medhurst}}$. However, this neglects perturbations from the environmental and the electrodes that, at minimum, likely break the assumption of top-bottom symmetry ($C_{t,1} \neq C_{t,2}$).

I may have found a better way to derive the tertiary model, and I hope to explore it in future work. Additionally, it might be interesting to see whether the magnetic scalar potential could construct an "inductance matrix" for magnetostatics similar to the capacitance matrix of electrostatics, and whether that matrix might have any relation to the dual case of negative inductance briefly explored above.

Appendix A: $ABCD$ solution for a uniform transmission line

Consider a one-dimensional transmission line with voltage $V(x,t)$ and current $I(x,t)$ at position x and time t along the line given by the Telegrapher's equations

$$\frac{\partial V(x,t)}{\partial x} = -\left(r + l\frac{\partial}{\partial t}\right) I(x,t) \tag{A1}$$

$$\frac{\partial I(x,t)}{\partial x} = -\left(g + c\frac{\partial}{\partial t}\right) V(x,t), \tag{A2}$$

with distributed series resistance r, series inductance l, shunt conductance g, and shunt capacitance c, all with units per length. Note that there is a sign freedom with current in lines. For this note, let positive current flow clockwise.

The following $ABCD$ form[10,11] provides a general way to write a time-harmonic solution with voltage and current that are both proportional to $e^{i\omega t}$ for positive ω,

$$\begin{pmatrix} V(x,t;\omega) \\ I(x,t;\omega) \end{pmatrix} = \begin{pmatrix} A(x;\omega) & B(x;\omega) \\ C(x;\omega) & D(x;\omega) \end{pmatrix} \begin{pmatrix} V(0,t;\omega) \\ I(0,t;\omega) \end{pmatrix}, \tag{A3}$$

where the matrix entries for our current convention are

$$A(x;\omega) = \cosh[\gamma(\omega)x] \tag{A4}$$

$$B(x;\omega) = -Z_0(\omega)\sinh[\gamma(\omega)x] \tag{A5}$$

$$C(x;\omega) = -Z_0^{-1}(\omega)\sinh[\gamma(\omega)x] \tag{A6}$$

$$D(x;\omega) = A(x,\omega). \tag{A7}$$

Here, the propagation "constant" is

$$\gamma(\omega) = \sqrt{z(\omega)y(\omega)} \tag{A8}$$

and the characteristic impedance is

$$Z_0(\omega) = \sqrt{z(\omega)/y(\omega)}, \tag{A9}$$

both in terms of a distributed series impedance z and a shunt admittance y of the line:

$$z(\omega) = r + i\omega l \tag{A10}$$

$$y(\omega) = g + i\omega c. \tag{A11}$$

You can verify that this $ABCD$ form is a solution by direct substitution. To help, note that for a time-harmonic solution, the Telegrapher's equations simplify to

$$\frac{\partial V(x,t;\omega)}{\partial x} = -z(\omega)I(x,t;\omega) \tag{A12}$$

$$\frac{\partial I(x,t;\omega)}{\partial x} = -y(\omega)V(x,t;\omega), \tag{A13}$$

and that after substitution, the derivatives follow from $d\cosh(ax)/dx = a\sinh(x)$ and $d\sinh(ax)/dx = a\cosh(x)$. For the lossless case of $r = g = 0$, $\gamma = i|\omega|\sqrt{lc}$ and $Z_0 = \sqrt{l/c}$.

Appendix B: ∇–Y transform

There's a well-known circuit transform to add or remove a node that goes by several names of the form "(wye, tee, or star)–(delta, triangle, pi, or mesh)" or vice versa, often written using symbols like Y–Δ. (Fun aside: Flipping the delta Δ to a del ∇ visually aligns the shared nodes, giving Y–∇ or ∇–Y.) Let's derive this transform, investigate if it can remove negative reactances, and show that it doesn't always preserve energy.

1. Derivation

Consider two circuits made from nodes identified by numbers n and impedances Z_{nm} that connect each pair of nodes n and m. Let the first be a "Delta" circuit made of only three nodes ($n = 1$, 2, and 3) and with only three finite impedances: Z_{12}, Z_{13}, and Z_{23}. [All other impedances are assumed open circuit (infinite).] Let the second be a "Wye" circuit have the same three "external" nodes, but adds a new "internal" 4th node and changes the impedances so that the only three finite impedances are Z_{14}, Z_{24}, and Z_{34}.

These two circuits are equivalent if the first three "external" nodes behave the same when connected to any external circuits, which occurs if the set $\{I_1, I_2, I_3\}$ of external currents into those nodes are identical for the same set $\{V_1, V_2, V_3\}$ of node voltages (and vice versa). As we'll derive below, this occurs if

$$Z_{14} = Z_{12}Z_{13}/Z_{(\Delta\Sigma)} = Z^2_{(\Delta\|)}/Z_{23} \tag{B1}$$

$$Z_{24} = Z_{12}Z_{23}/Z_{(\Delta\Sigma)} = Z^2_{(\Delta\|)}/Z_{13} \tag{B2}$$

$$Z_{34} = Z_{13}Z_{23}/Z_{(\Delta\Sigma)} = Z^2_{(\Delta\|)}/Z_{12}, \tag{B3}$$

where the total (or loop) sum of the Delta circuit's impedances in series is

$$Z_{(\Delta\Sigma)} = Z_{12} + Z_{13} + Z_{23} = Z_{12}Z_{13}Z_{23}/Z^2_{(\Delta\|)} \tag{B4}$$

and the parallel sum of all pairwise products of impedances in the Delta circuit is

$$Z^2_{(\Delta\|)} = \frac{1}{\frac{1}{Z_{12}Z_{13}} + \frac{1}{Z_{12}Z_{23}} + \frac{1}{Z_{13}Z_{23}}}. \tag{B5}$$

These equations define a way to transform a Delta to a Wye circuit. To transform from a Wye to a Delta circuit, the inverse equations are

$$Z_{12} = Z_{14}Z_{24}/Z_{(Y\|)} = Z^2_{(Y\Sigma)}/Z_{34} \tag{B6}$$

$$Z_{13} = Z_{14}Z_{34}/Z_{(Y\|)} = Z^2_{(Y\Sigma)}/Z_{24} \tag{B7}$$

$$Z_{23} = Z_{24}Z_{34}/Z_{(Y\|)} = Z^2_{(Y\Sigma)}/Z_{14}, \tag{B8}$$

where the total sum of the Wye circuit's impedances in parallel is

$$Z_{(Y\|)} = \frac{1}{\frac{1}{Z_{14}} + \frac{1}{Z_{24}} + \frac{1}{Z_{34}}} = \frac{Z_{14}Z_{24}Z_{34}}{Z^2_{(Y\Sigma)}} \tag{B9}$$

and the series sum of all pairwise products of impedances in the Wye circuit is

$$Z^2_{(Y\Sigma)} = Z_{14}Z_{24} + Z_{14}Z_{34} + Z_{24}Z_{34}. \tag{B10}$$

To derive these transforms, let the I_{nm} denote the current flowing from node n to node m, such that $I_{nm} = -I_{mn}$, let $V_{nm} = V_n - V_m = -V_{mn}$ denote the difference in node voltages, and let $Z_{nm} = Z_{mn}$. Starting with the Delta circuit, the total external current flowing into the first node is

$$I_1 = I_{12} + I_{13}. \tag{B11}$$

Each of those internal currents is of the form

$$I_{12} = V_{12}/Z_{12}. \tag{B12}$$

Combining gives

$$I_1 = V_1\left(\frac{1}{Z_{12}} + \frac{1}{Z_{13}}\right) - V_2\left(\frac{1}{Z_{12}}\right) - V_3\left(\frac{1}{Z_{13}}\right), \tag{B13}$$

and similar for the other external nodes. Next, for the Wye circuit, the same current is

$$I_1 = I_{14} = V_{14}/Z_{14}, \tag{B14}$$

and likewise for the other external nodes. The total current at the internal node must balance, so

$$I_{14} + I_{24} + I_{34} = 0. \tag{B15}$$

Combining gives the internal node voltage,

$$V_4 = Z_{(Y\|)}\left(V_1/Z_{14} + V_2/Z_{24} + V_3/Z_{34}\right), \tag{B16}$$

which in turn gives the first external current as

$$I_1 = V_1\left(\frac{1}{Z_{14}}\right)\left(1 - \frac{Z_{(Y\|)}}{Z_{14}}\right) - V_2\left(\frac{Z_{(Y\|)}}{Z_{14}Z_{24}}\right) - V_3\left(\frac{Z_{(Y\|)}}{Z_{14}Z_{34}}\right), \tag{B17}$$

and similar for the other external nodes. Therefore, the two circuits have identical I_1 for the same set $\{V_1, V_2, V_3\}$ of external node voltages if the coefficients on each voltage match. The V_2 and V_3 coefficients immediately lead to the Wye-to-Delta transform results for Z_{12} and Z_{13}, and the V_1 coefficients agree after substituting those results. The same would occurs if we repeated this for I_2 and I_3, completing the Wye-to-Delta transform derivation. To go the other direction, the V_1 coefficients can be used to solve for $Z_{12}Z_{13}$, givining the results for Z_{14}. Repeating for the other currents then completes the Delta-to-Wye transform derivation.

Let's consider some special cases: If one of the Delta connections is open circuit, then the Wye circuit has one short circuit (e.g., if $|Z_{12}| = \infty$, then $Z_{34} = 0$, $Z_{14} = Z_{13}$, and $Z_{24} = Z_{23}$). If one of the Delta connections is short circuit, then the Wye circuit has two short circuits (e.g., if $Z_{12} = 0$, then $Z_{14} = Z_{24} = 0$ and $Z_{34} = 1/[1/Z_{13} + 1/Z_{23}]$). If one of the Wye connections is open circuit, then the Delta circuit has two open circuits (e.g., if $|Z_{14}| = \infty$, then $|Z_{12}| = \infty$, $|Z_{13}| = \infty$, and $Z_{23} = Z_{24} + Z_{34}$).

2. Failure to remove negativity

Next, let's consider signs: Suppose you have a Delta circuit, and one of its impedances is negative. Then transforming to a Wye circuit will lead to either one or two negative impedances, depending on the sign of $Z_{(\Delta\Sigma)}$. Likewise, if you have a Wye circuit with a single negative impedance, then transforming leads to a Delta circuit with one or two negative impedances, depending on the sign of $Z_{(Y\|)}$. Therefore, it seems like negative impedances are hard to remove. Thus, for example, if you have a negative capacitance between two nodes in a network of capacitors, you likely can't transform it away in general.

3. Failure to preserve energy

As derived, the ∇–Y transform preserves impedance, so the circuits it generates will equivalently model the same steady-state (or equilibrium) voltages and currents. However, what about stored energy? Here's a counter example showing that the ∇–Y transform is not guaranteed to preserve stored energy.

Consider a Wye circuit made of identical inductances: $Z_{14} = Z_{24} = Z_{34} = Z = i\omega L$. Let the external node voltages be $V_1 = 0$, $V_2 = 1$, and $V_3 = 2$, so that the currents are $I_1 = -I_3 = -1/Z$ and $I_2 = 0$ (thus $V_2 = V_4$). The stored energy is $L(|I_1|^2 + |I_3|^2)/2 = L/|Z|^2 = 1/(L\omega^2)$.

Transforming to a Delta circuit gives impedances $Z_{12} = Z_{13} = Z_{23} = Z' = 3Z$ and internal currents $I_{13} = -2/Z'$ and $I_{12} = I_{23} = -1/Z'$. The stored energy is $L(|I_{12}|^2 + |I_{13}|^2 + |I_{23}|^2)/2 = 3L/|Z'|^2 = L/(3|Z|^2) = 1/(3L\omega^2) \neq 1/(L\omega^2)$. This disagrees with the energy stored by the Wye circuit, demonstrating that the ∇–Y transform does not preserve stored energy in general.

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