Affine transformations and 2D Fourier transforms

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https://bartmcguyer.com/notes/note-8-AffineTheorem.pdf

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TL;DR: Concise formulation of handy properties of two-dimensional (2D) Fourier transforms under linear coordinate transformations.

Linear coordinate transformations that preserve collinearity (straight lines stay straight, parallel lines stay parallel) and proportions on lines (midpoints stay midpoints) but not necessarily angles or lengths are known as affine transformations.[1] They include translations, rotations, reflections, scalings, shears, etc., as well as their combinations, forming an affine group. They're important to computer graphics,[2,3] fractals (self affinity), and moirés.[4,5]

Affine transformations seem to be the most general type of transformation with convenient Fourier-transform properties. This note derives three versions of the so-called affine theorem.[2,3] Together, they describe how affine transformations are related between the image and frequency domains of a 2D Fourier transform. They're particularly useful because they concisely combine some well-known properties of Fourier transforms.

A summary of these results are provided in the Appendix of Ref. 5.

CONTENTS

I. AFFINE THEOREMS

The next sections present three versions of the affine theorem followed by their derivation. The first is the traditional version that gives the effect in the frequency domain of an affine transformation applied in the image domain. The second is the converse of the first, giving the effect in the image domain of an affine transformation applied in the frequency domain. The third is a generalization that combines the previous two. Afterwards, there is a derivation of the third version that includes the first and second versions as special cases.

To begin, we must choose a Fourier-transform convention. Let's use

$$G(u,v) = \mathcal{F}[g] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-2\pi i(ux+vy)}dxdy \tag{1}$$

$$g(x,y) = \mathcal{F}^{-1}[G] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(u,v)e^{2\pi i(ux+vy)}dudv \tag{2}$$

for the transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$, following Refs. 2 and 3. Here and subsequently, let's use upper-case functions [i.e., $G(u,v)$] to denote the 2D Fourier transforms of lower-case functions [i.e., $g(x,y)$], and call two such functions a Fourier-transform pair. Let's refer to the (x,y) domain as the image domain and the (u,v) domain as the frequency domain. The results below can be readily adapted to other conventions (see below) and to more than 2D.

A. Transformation in the image domain

Consider the following affine transformation of the coordinates (x,y) to (x',y'):

$$\begin{cases} x' = ax + by + x_0 \\ y' = cx + dy + y_0. \end{cases} \tag{3}$$

We can rewrite this in matrix notation as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \tag{4}$$

or even more compactly as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{x_0}. \tag{5}$$

The inverse of this affine transformation is

$$\mathbf{x} = \mathbf{A}^{-1}(\mathbf{x}' - \mathbf{x_0}). \tag{6}$$

For reference, the matrix $\mathbf{A}$, its inverse $\mathbf{A}^{-1}$, and its inverse transpose $\mathbf{A}^{-T}$ are

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \mathbf{A}^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \text{and} \qquad \mathbf{A}^{-T} = \frac{1}{|A|}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}, \tag{7}$$

respectively, where the determinant

$$|A| = \det(A) = ad - bc \tag{8}$$

must be nonzero, $|A| \neq 0$, for the inverse and inverse transpose matrices to exist. Note that $\mathbf{A}^{-1}$ and $\mathbf{A}^{-T}$ are also affine transformations.

Affine theorem #1 (traditional, image domain): If $f(x,y)$ and $F(u,v)$ are a Fourier-transform pair, then the following functions are also a Fourier-transform pair:

$$g_1(x,y) = f(ax + bx + x_0, cx + dy + y_0) \tag{9}$$

$$G_1(u,v) = \frac{1}{|\det(A)|}\exp\left\{\frac{2\pi i}{|A|}\left[(dx_0 - by_0)u + (ay_0 - cx_0)v\right]\right\} F\left(\frac{du - cv}{|A|}, \frac{av - bu}{|A|}\right). \tag{10}$$

We can rewrite this in matrix notation as

$$g_1(\mathbf{x}) = f(\mathbf{x}') = f(\mathbf{A}\mathbf{x} + \mathbf{x_0}) \tag{11}$$

$$G_1(\mathbf{u}) = \frac{1}{|\det(A)|} e^{2\pi i\, \mathbf{x_0}^T \mathbf{A}^{-T}\mathbf{u}} F\left(\mathbf{A}^{-T}\mathbf{u}\right). \tag{12}$$

This is the traditional affine theorem.[2,3] It shows that the transformation $\mathbf{A}\mathbf{x} + \mathbf{x_0}$ in the image domain induces a corresponding affine transformation $\mathbf{A}^{-T}\mathbf{u}$ in the frequency domain as well as a linear phase modulation $e^{2\pi i\, \mathbf{x_0}^T \mathbf{A}^{-T}\mathbf{u}}$ and an amplitude scaling $1/\left|\det(A)\right|$. Note, however, that the corresponding transformation doesn't include translations.

B. Transformation in the frequency domain

Consider the following affine transformation of the coordinates (u, v) to (u', v'):

$$\begin{cases} u' = \alpha u + \beta v + u_0 \\ v' = \gamma u + \delta v + v_0. \end{cases} \tag{13}$$

As before, we can rewrite this in matrix notation as

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \tag{14}$$

or even more compactly as

$$\mathbf{u}' = \mathbf{B}\mathbf{u} + \mathbf{u_0}. \tag{15}$$

The inverse of this affine transformation is

$$\mathbf{u} = \mathbf{B}^{-1}(\mathbf{u}' - \mathbf{u_0}). \tag{16}$$

For reference, the matrix $\mathbf{B}$, its inverse $\mathbf{B}^{-1}$, and its inverse transpose $\mathbf{B}^{-T}$ are

$$\mathbf{B} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \mathbf{B}^{-1} = \frac{1}{|B|}\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}, \quad \text{and} \qquad \mathbf{B}^{-T} = \frac{1}{|B|}\begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix}, \tag{17}$$

respectively, where the determinant

$$|B| = \det(B) = \alpha\delta - \beta\gamma \tag{18}$$

must be nonzero, $|B| \neq 0$, for the inverse and inverse transpose matrices to exist. Again, note that $\mathbf{B}^{-1}$ and $\mathbf{B}^{-T}$ are also affine transformations.

Affine theorem #2 (frequency domain): If $f(x, y)$ and $F(u, v)$ are a Fourier-transform pair, then the following functions are also a Fourier-transform pair:

$$G_2(u, v) = F(\alpha u + \beta v + u_0, \gamma u + \delta v + v_0) \tag{19}$$

$$g_2(x, y) = \frac{1}{|\det(B)|} \exp\left\{\frac{-2\pi i}{|B|}\left[(\delta u_0 - \beta v_0)x + (\alpha v_0 - \gamma u_0)y\right]\right\} f\left(\frac{\delta x - \gamma y}{|B|}, \frac{\alpha y - \beta x}{|B|}\right). \tag{20}$$

We can rewrite this in matrix notation as

$$G_2(\mathbf{u}) = F(\mathbf{u}') = F(\mathbf{B}\mathbf{u} + \mathbf{u_0}) \tag{21}$$

$$g_2(\mathbf{x}) = \frac{1}{|\det(B)|} e^{-2\pi i\, \mathbf{u_0}^T \mathbf{B}^{-T}\mathbf{x}} f\left(\mathbf{B}^{-T}\mathbf{x}\right). \tag{22}$$

This second version is nearly identical to the traditional affine theorem, but with the image and frequency domains swapped. As before, the corresponding affine transformation, this time induced in the image domain, does not include translations.

C. Corresponding transformations in both the image and frequency domains

Affine theorem #3 (image and frequency domains): If $f(x, y)$ and $F(u, v)$ are a Fourier-transform pair, the transformations defined previously satisfy

$$\mathbf{A} = \mathbf{B}^{-T} \quad \left(\text{or equivalently,} \quad \mathbf{B} = \mathbf{A}^{-T}\right), \tag{23}$$

and their determinants are nonzero, $|A| \neq 0$ and $|B| \neq 0$, then the following functions are also a Fourier-transform pair:

$$g_3(\mathbf{x}) = e^{-2\pi i\, \mathbf{u_0}\cdot\mathbf{x}'} f(\mathbf{x}') \tag{24}$$

$$G_3(\mathbf{u}) = \left(|\det(B)|\, e^{-2\pi i\, \mathbf{x_0}\cdot\mathbf{u_0}}\right) e^{2\pi i\, \mathbf{x_0}\cdot\mathbf{u}'} F(\mathbf{u}'). \tag{25}$$

This third version includes a new phase factor $e^{2\pi i\, \mathbf{x_0}\cdot\mathbf{u_0}}$ that is significant when there are simultaneous translations in both the image and frequency domains, so is absent in the first and second versions. It recovers the previous two versions when either one of these translations is absent: If $\mathbf{u_0} = 0$, then $g_3(x, y) = g_1(x, y)$ and $G_3(u, v) = G_1(u, v)$; If $\mathbf{x_0} = 0$, then $g_3(x, y) = |\det(A)|\, g_2(x, y)$ and $G_3(u, v) = |\det(A)|\, G_2(u, v)$. Note that the amplitude scaling $|\det(A)|$ and the new phase factor $e^{-2\pi i\, \mathbf{x_0}\cdot\mathbf{u_0}}$ inside the parenthesis in (25) can be moved between the functions $g_3(x, y)$ and $G_3(u, v)$ by redefining them. The linear phase gradients in both functions are equivalent to the so-called shift theorem of Fourier transforms.[2]

From (23), the coefficients of corresponding affine transformations (7) and (17) satisfy

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{|B|}\begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix}, \quad \text{or equivalently,} \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{1}{|A|}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}, \tag{26}$$

where the determinants

$$|A| = ad - bc = \frac{1}{|B|} = \frac{1}{\alpha\delta - \beta\gamma}. \tag{27}$$

However, there's no relationship required between the translation coefficients x_0, y_0, u_0, and v_0. Table I shows examples of matrices $\mathbf{A}$ and $\mathbf{B}$ satisfying (23).

TABLE I. Examples of corresponding affine-transformation matrices arranged vertically.

	Identity/Reflection	Scaling	Rotation	Shear
$\mathbf{A} = \mathbf{B}^{-T}$:	$\begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}$	$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$	$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$
$\mathbf{B} = \mathbf{A}^{-T}$:	$\begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1/s_x & 0 \\ 0 & 1/s_y \end{pmatrix}$	$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ -m & 1 \end{pmatrix}$

D. Derivation

We can derive the third version of the affine theorem, which includes the first and second versions as special cases, as follows. Using (24) with (1), the function

$$G_3(u,v) = \mathcal{F}[g_3] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i(ux+vy+\mathbf{u_0}\cdot\mathbf{x}')} f(x',y')dxdy. \tag{28}$$

We can change the variables of integration using

$$dxdy = \left|\frac{\partial(x,y)}{\partial(x',y')}\right| dx'dy' = \left|\det\left(A^{-1}\right)\right| dx'dy' = \left|\det(B)\right| dx'dy' \tag{29}$$

which gives

$$G_3(u,v) = \left|\det(B)\right| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i(ux+vy+\mathbf{u_0}\cdot\mathbf{x}')} f(x',y')dx'dy'. \tag{30}$$

The argument of the exponent can be rewritten in terms of transformed variables as

$$\begin{aligned} ux + vy &= \mathbf{u}^T\mathbf{x} && (31)\\ &= [B^{-1}(\mathbf{u}' - \mathbf{u_0})]^T A^{-1}(\mathbf{x}' - \mathbf{x_0}) && (32)\\ &= (\mathbf{u}' - \mathbf{u_0})^T B^{-T} A^{-1}(\mathbf{x}' - \mathbf{x_0}) && (33)\\ &= (\mathbf{u}' - \mathbf{u_0})^T(\mathbf{x}' - \mathbf{x_0}) && (34)\\ &= u'x' + v'y' - \mathbf{x_0}\cdot\mathbf{u}' - \mathbf{u_0}\cdot\mathbf{x}' + \mathbf{x_0}\cdot\mathbf{u_0} && (35) \end{aligned}$$

Using this substitution in (30) gives

$$\begin{aligned} G_3(u,v) &= \left|\det(B)\right| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i(u'x'+v'y'-\mathbf{x_0}\cdot\mathbf{u}'-\mathbf{u_0}\cdot\mathbf{x_0})} f(x',y')dx'dy' && (36)\\ &= \left|\det(B)\right| e^{2\pi i(\mathbf{x_0}\cdot\mathbf{u}'-\mathbf{u_0}\cdot\mathbf{x_0})} F(u',v'), && (37) \end{aligned}$$

which recovers (25).

E. Adapting to different Fourier transform conventions

We can adapt these results to different conventions as follows. Consider the following general Fourier transform and its inverse

$$G(u,v) = \mathcal{F}_{(C,D)}[g(x,y)](u,v) = CD \iint g(x,y) e^{-i2\pi D(xu+yv)} dxdy \tag{38}$$

$$g(x,y) = \mathcal{F}^{-1}_{(C,D)}[G(u,v)](x,y) = \frac{D}{C} \iint G(u,v) e^{i2\pi D(ux+vy)} dudv. \tag{39}$$

Different conventions correspond to different values of the coefficients C and D. For example, (1) and (2) used everywhere above correspond to $C = D = 1$. Adapting the derivation above, we can rephrase affine theorem #3 as follows.

Affine theorem #4 (image and frequency domains): If $f(x,y)$ and $F(u,v)$ are a Fourier-transform pair satisfying (38) and (39), the transformations defined previously satisfy

$$\mathbf{A} = \mathbf{B}^{-T} \quad \big(\text{or equivalently,} \quad \mathbf{B} = \mathbf{A}^{-T}\big), \tag{40}$$

and their determinants are nonzero, $|A| \neq 0$ and $|B| \neq 0$, then the following functions are also a Fourier-transform pair satisfying (38) and (39):

$$g_4(\mathbf{x}) = e^{-i2\pi D\, \mathbf{u_0} \cdot \mathbf{x}'} f(\mathbf{x}') \tag{41}$$

$$G_4(\mathbf{u}) = \big(|\det(B)|\, e^{-i2\pi D\, \mathbf{x_0} \cdot \mathbf{u_0}}\big)\, e^{i2\pi D\, \mathbf{x_0} \cdot \mathbf{u}'} F(\mathbf{u}'). \tag{42}$$

II. COMBINING AFFINE TRANSFORMATIONS

A convenient way to synthesize affine transformations is to combine them using the following modified notation: Rewriting (3) with an additional dimension as

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & x_0 \\ c & d & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}, \tag{43}$$

a single 3-by-3 matrix now includes the effects of the both matrix $\mathbf{A}$ and vector $\mathbf{x_0}$. Such 3-by-3 matrices can then be multiplied to create new transformations. For example, we can synthesize an offset rotation about an arbitrary point (x_0, y_0) by multiplying the 3-by-3 matrices for a translation, rotation, and inverse translation, giving

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{pmatrix} \\ &\qquad = \begin{pmatrix} \cos(\theta) & \sin(\theta) & [1-\cos(\theta)]x_0 - \sin(\theta) y_0 \\ -\sin(\theta) & \cos(\theta) & \sin(\theta) x_0 + [1-\cos(\theta)] y_0 \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned} \tag{44}$$

which is a rotation about the origin with a rotation-dependent translation.

REFERENCES

[1]Wolfram Mathworld has a good introduction to affine transformations:
`http://mathworld.wolfram.com/AffineTransformation.html`

[2]Ronald N. Bracewell, *Fourier Analysis and Imaging*, Springer (2003). For the affine theorem, see p. 159–161.

[3]R. N. Bracewell, K.-Y. Chang, A. K. Jha and Y.-H. Wang, "Affine theorem for two-dimensional Fourier transform," Electronics Letters, Vol. 29, no. 3, p. 304 (1993). DOI: 10.1049/el:19930207

[4]Isaac Amidror, *The Theory of the Moiré Phenomenon, Volume I: Periodic Layers*, 2nd ed., Springer (2009). (Partially available online via Google Books.) For the affine theorem, see p. 258–259.

[5]B. H. McGuyer and Qi Tang, "Connection between antennas, beam steering, and the moiré effect," arXiv:2107.05571 (2021).