Nonharmonic Fourier series

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https://bartmcguyer.com/notes/note-10-NonharmonicFourier.pdf

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TL;DR: Examples of complete orthogonal systems of sine and cosine functions that do not have a harmonic relationship.

This note presents some unusual complete sets of orthogonal basis functions that are useful in electrical problems with transmission lines and impedance matching.[1] These sets resemble Fourier series because they are made of sine and cosine functions. However, their wavenumbers (or eigenvalues) do not follow a harmonic progression, so they are known as nonharmonic Fourier series and sit at an interesting boundary between harmonic Fourier series and generalized Fourier series (sets of special functions). These sets can't be novel, but it seems difficult to find documentation about them.[2]

CONTENTS

I. SETUP

Consider the ordinary differential equation

$$\frac{d^2}{dx^2}\, y(x) + \lambda\, y(x) = 0 \tag{1}$$

for the function $y(x)$ with the domain $x \in [x_1, x_2]$. This could be a harmonic oscillator or the spatial portion of a 1D wave equation or the Telegrapher's equations for a transmission line (see Section V). Let its boundary conditions be of a mixed (Robin or impedance) type,[3]

$$c_1\, y(x_1) + c_2\, y'(x_1) = 0 \tag{2}$$

$$c_3\, y(x_2) + c_4\, y'(x_2) = 0 \tag{3}$$

where c_n are constant coefficients and a prime denotes differentiation (d/dx).

In general, the solutions are of the form $y(x) \propto \cos(kx + \phi)$, with the allowed values of k and ϕ—and resulting orthogonality—depending on the boundary conditions. For typical open or closed boundary conditions with each end having only one nonzero c_n, this leads to conventional Fourier series with quarter-wave or half-wave harmonic progressions for k. But, what about mixed cases in between?

Fortunately, this is a "regular" Sturm-Liouville problem,[4] so it is guaranteed to have a set of orthogonal eigenfunctions with positive eigenvalues λ that forms a complete basis over the domain. The next sections present some examples of basis sets that are controlled by a single parameter, which vary from one harmonic limit to another with anharmonic progressions in between.

Here are some overlap integrals we'll need:

$$A_H(a,b) = \int_0^H \sin(ax)\sin(bx)dx = \begin{cases} 0 & a=b=0 \\ \frac{H}{2} - \frac{\sin(2aH)}{4a} & a=b\neq 0 \\ \frac{a\cos(aH)\sin(bH)-b\cos(bH)\sin(aH)}{b^2-a^2} & a\neq b, a \text{ or } b\neq 0 \end{cases} \tag{4}$$

$$B_H(a,b) = \int_0^H \cos(ax)\cos(bx)dx = \begin{cases} H & a=b=0 \\ \frac{H}{2} + \frac{\sin(2aH)}{4a} & a=b\neq 0 \\ \frac{a\sin(aH)\cos(bH)-b\sin(bH)\cos(aH)}{a^2-b^2} & a\neq b, a \text{ or } b\neq 0 \end{cases} \tag{5}$$

$$C_H(a,b) = \int_0^H \sin(ax)\cos(bx)dx = \begin{cases} 0 & a=b=0 \\ \frac{\sin(aH)^2}{2a} & a=b\neq 0 \\ \frac{a-a\cos(aH)\cos(bH)-b\sin(bH)\sin(aH)}{a^2-b^2} & a\neq b, a \text{ or } b\neq 0 \end{cases} \tag{6}$$

II. SINE SERIES

Consider the range $x \in [0, H]$ with the boundary conditions

$$y(0) = 0 \tag{7}$$

$$\sqrt{z/H}\, y(H) - \sqrt{H/z}\, y'(H) = 0 \tag{8}$$

where the positive, real-valued control parameter $z \in [0, \infty]$.

From the first boundary condition (7), the solutions must have the form

$$y(x) \propto \sin(kx) \tag{9}$$

with the allowed values of k determined by the second boundary condition.

As setup, the second boundary condition (8) becomes $y'(H) = 0$ in the limit of $z \longrightarrow 0$, leading to a quarter-wave Fourier sine series with the allowed values $k_n = (2n-1)\pi/(2H) = \pi/(2H), 3\pi/(2H), 5\pi/(2H), \ldots$ for integer $n = 1, 2, 3, \ldots$ Likewise, it becomes $y(H) = 0$ in the limit of $z \longrightarrow \infty$, leading to a half-wave Fourier sine series with the allowed values

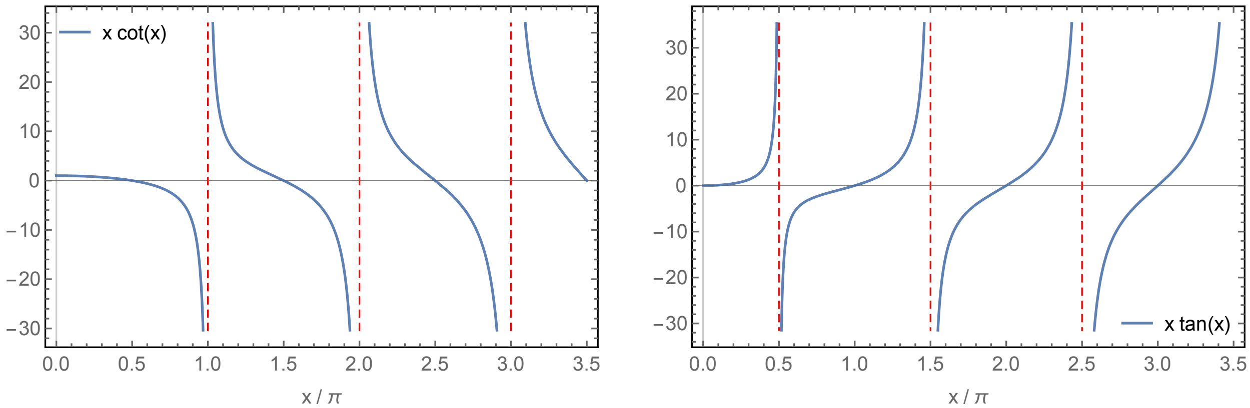

FIG. 1. Transcendental equations from the boundary conditions for the sine (left) and cosine (right) series. Note that $\lim_{x \longrightarrow 0} x\cot(x) = 1$ and $x\tan(x) = 0$.

$k_n = (n-1)\pi/H = \pi/H, 2\pi/H, 3\pi/H, \ldots$ for $n = 2, 3, 4, \ldots$ ($n = 1$ is excluded to align the index for quarter- and half-wave series—see below.)

For finite $z > 0$, the second boundary condition (8) is equivalent to

$$kH\cot(kH) = z. \tag{10}$$

The allowed values of k are given by the solutions of this transcendental equation, and correspond to the intersections between a horizontal line with value z and the curves for $x\cot(x)$ shown in Fig. 1. Note that the limiting behaviors of $z \longrightarrow 0$ and ∞ are also captured by this transcendental equation.

We can construct an orthonormal basis as the set of functions

$$s_n^{\{z\}}(x) = \frac{\sin(k_n^{\{z\}}x)}{\sqrt{A_H\left(k_n^{\{z\}}, k_n^{\{z\}}\right)}} \tag{11}$$

where $k_n^{\{z\}}$ denotes the n-th solution of the transcendental equation. (Note that the limiting case of $k_1^{\{\infty\}} \longrightarrow 0$ is excluded because $\sin(0) = 0$.) These basis functions are orthogonal because, for solutions of the second boundary condition, $A_H\left(k_n^{\{z\}}, k_m^{\{z\}}\right) = 0$ for $n \neq m$.

Figure 2 shows the first few basis functions for different values of z, highlighting a transition from quarter-wave to half-wave harmonic limits. The numbering connects the functions between different cases, with the lowest ($n = 1$) becoming absent for $z \geq 1$.

III. COSINE SERIES

Consider the range $x \in [0, H]$ with the boundary conditions

$$y'(0) = 0 \tag{12}$$

$$\sqrt{z/H}\, y(H) - \sqrt{H/z}\, y'(H) = 0 \tag{13}$$

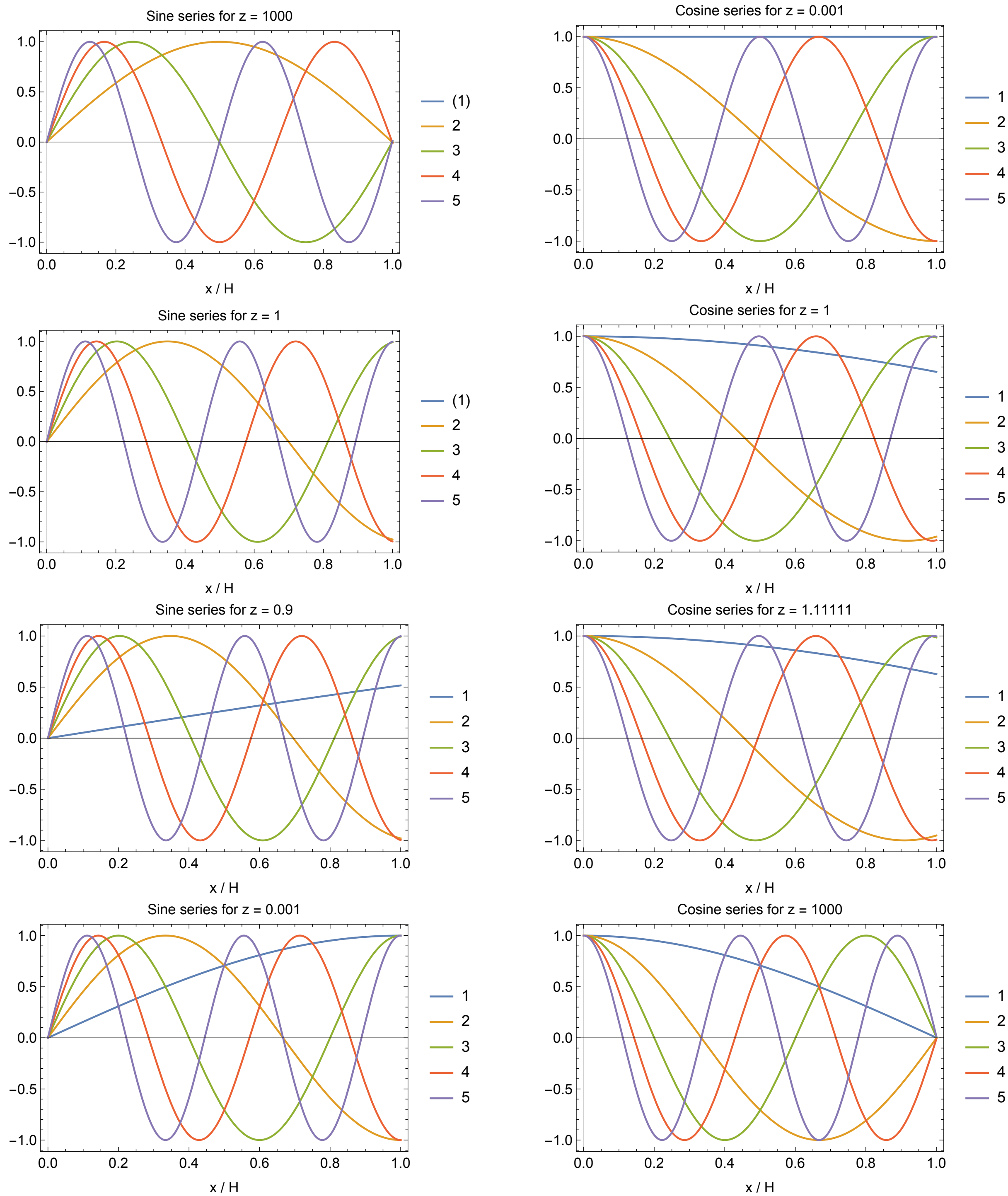

FIG. 2. First few basis functions for the sine (left) and cosine (right) series for different values of their control parameters z, transitioning from near a half-wave limit (top) to near a quarter-wave limit (bottom). The sine series loses its $n = 1$ function as z passes through 1 from below. The basis functions are not normalized for clarity. Note that the quarter-wave limits (bottom) are mirror images of each other (left to right).

where the positive, real-valued control parameter $z \in [0, \infty]$.

The only change from before is that the first boundary condition (12) is modified so that the solutions now have the form

$$y(x) \propto \cos(kx). \tag{14}$$

This reverses the limiting behavior for the second boundary condition (13), which now leads to a half-wave Fourier cosine series for $z \longrightarrow 0$ and a quarter-wave Fourier cosine series for $z \longrightarrow \infty$. Unlike before, there is no abrupt removal of a function as z varies.

For finite $z > 0$, the second boundary condition (13) is equivalent to

$$kH \tan(kH) = z. \tag{15}$$

As before, the allowed values of k are given by the solutions of this similar transcendental equation, and correspond to the intersections between a horizontal line with value z and the curves for $x \tan(x)$ shown in Fig. 1. Note that the limiting behaviors of $z \longrightarrow 0$ and ∞ are again captured by the transcendental equation.

We can construct an orthonormal basis as the set of functions

$$s_n^{\{z\}}(x) = \frac{\cos(k_n^{\{z\}} x)}{\sqrt{B_H\left(k_n^{\{z\}}, k_n^{\{z\}}\right)}} \tag{16}$$

where $k_n^{\{z\}}$ denotes the n-th solution of the transcendental equation. (Unlike before, the limiting case of $k_1^{\{\infty\}} \longrightarrow 0$ with $s_1^{\{\infty\}} = 1/\sqrt{H}$ is included because $\cos(0) = 1 \neq 0$.) Once again, these basis functions are orthogonal because, for solutions of the second boundary condition, $B_H\left(k_n^{\{z\}}, k_m^{\{z\}}\right) = 0$ for $n \neq m$.

Figure 2 shows the first few basis functions for different values of z, highlighting a transition from quarter-wave to half-wave harmonic limits. The numbering connects the functions between different cases.

IV. DISCUSSION

These series can be readily extended to negative z with similar results. For the sine series, this gets rid of the disappearance of the null-able $n = 1$ function about $z = 1$. For the cosine series, this causes the constant $n = 1$ function present with $z = 0$ to disappear for $z < 0$.

Interestingly, the sine and cosine series given above are not mutually orthogonal in general, in contrast with typical "biorthogonal" harmonic Fourier series. In fact, $C_H(a, b)$ suggests that there doesn't seem to be a way to pair anharmonic sine and cosine series to be mutually orthogonal. One way to produce a more general series might be to have mixed boundary conditions at both ends, which may produce a series with effectively both sine and cosine terms that are mutually orthogonal. For more on orthogonality, see the next section.

There is a similarity between these results and a "stretched" harmonic Fourier series that is complete over a larger domain than needed. Superficially, starting at the quarter-wave limit, varying z appears to stretch the domain of each function in these series given above beyond $[0, H]$, but does so differently for each function. This stretching only aligns again harmonically in the limiting case of $z \longrightarrow \infty$, except for $n = 0$. If you were to instead construct a harmonic Fourier series over a larger domain $[0, H']$ with $H' > H$, the results would look roughly similar. However, the basis functions would no longer be orthogonal on the original domain $[0, H]$, but instead would be "over complete" on that domain. This topic is important to and discussed in Ref. 1.

V. CONNECTION WITH TRANSMISSION LINES

We can make a connection with standing-wave resonances on transmission lines as follows. Consider a one-dimensional line with voltage $V(x,t)$ and current $I(x,t)$ at position x and time t along the line given by the Telegrapher's equations

$$\frac{\partial V}{\partial x} = -\left(r + l\frac{\partial}{\partial t}\right) I \tag{17}$$

$$\frac{\partial I}{\partial x} = -\left(g + c\frac{\partial}{\partial t}\right) V, \tag{18}$$

with series resistance r, series inductance l, shunt conductance g, and shunt capacitance c per length. By taking the spatial derivative of one of the equations, we can combine the equations as either

$$\left[\frac{\partial^2}{\partial x^2} - \left(r + l\frac{\partial}{\partial t}\right)\left(g + c\frac{\partial}{\partial t}\right)\right] V(x,t) = 0 \tag{19}$$

or another equation of the same form for $I(x,t)$. If we try solutions of the form $V(x,t) = X_v(x)T_v(t)$ and $I(x,t) = X_i(x)T_i(t)$, then these equations separate to

$$X_v'' + \lambda_v X_v = 0 \tag{20}$$

$$X_i'' + \lambda_i X_i = 0 \tag{21}$$

$$\left[\lambda_v^2 + \left(r + l\frac{\partial}{\partial t}\right)\left(g + c\frac{\partial}{\partial t}\right)\right] T_v = 0 \tag{22}$$

$$\left[\lambda_i^2 + \left(r + l\frac{\partial}{\partial t}\right)\left(g + c\frac{\partial}{\partial t}\right)\right] T_i = 0, \tag{23}$$

where primes denote differentiation. The first two equations show that the spatial dependence of the current and the voltage are both of the same form as (1).

Often, the ends of a line are terminated by other circuits described by impedances. This leads to boundary conditions of the form

$$V(x_n, t) = (-1)^n Z_n I(x_n, t) \tag{24}$$

for the ends $x_n = x_1$ and x_2, where Z_n is the terminating impedance at x_n. The sign factor accounts for the directon of the current $I(x,t)$ being reversed at x_1.

Going back to the Telegrapher's equations, for standing-wave resonances we are interested in damped oscillatory solutions of the form $T_v(t)$ and $T_i(t) \propto e^{st}$ with Laplace frequency $s = i\omega - \Gamma$. Then (17) and (18) can be re-arrange as

$$\frac{\partial V}{\partial x} = -\left(r + sl\right) I \tag{25}$$

$$\frac{\partial I}{\partial x} = -\left(g + sc\right) V. \tag{26}$$

Either (25) or (26) can be used to replace I with $\partial V/\partial x$ or V with $\partial I/\partial x$ in (24), making it of the same form as the mixed boundary conditions in (2) and (3). Therefore, the basis functions either for voltage or for current can be nonharmonic Fourier series. Note that the impedance usually also depends on s, as in $Z_n = Z_n(s)$, so the orthonormal basis for voltage or current depends on the value of s in general.

For a worked example of a line shorted to ground on one end and terminated by a capacitor on the other end, which is a toy model of an ideal Tesla coil, please see Ref. 1. For a more complicated example of nonharmonic Fourier series, please see Ref. 5, which adds a capacitive tap to such a line to form a toy model for a Tesla magnifier.

A. Nonorthogonality and energy conservation

Once curious consequence that typically comes with nonharmonic Fourier on transmission lines is that one or both of the series expansions for voltage $V(x,t)$ and current $I(x,t)$ are no longer orthogonal on the line. For example, if a line is terminated by a capacitor at $x = H$, then we would recover the orthogonal, nonharmonic cosine series described above for the current in the line.[1] However, the corresponding sine series for the voltage that follows from the telegrapher's equations is not an orthogonal series.

Physically, this nonorthogonality makes sense from energy conservation and normal modes as follows. Consider the instantaneous total energy stored by a lossless uniform line,

$$U(t) = \frac{1}{2}\int_0^H \left(lI(x,t)^2 + cV(x,t)^2\right) dx, \tag{27}$$

for real-valued current and voltage. Note that this energy is conserved internally in the line, but may change if work is done at the boundaries of the line,

$$\frac{dU(t)}{dt} = V(0,t)I(0,t) - V(H,t)I(H,t). \tag{28}$$

(Note that positive $I(x,t)$ flows towards increasing x, which points inward at $x = 0$ and outward at $x = H$.) This follows from the telegrapher's equations and integration by parts,

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\int_0^H cV(x,t)^2 dx\right) &= \int_0^H cV(x,t)\frac{dV(x,t)}{dt}dx = -\int_0^H V(x,t)\frac{dI(x,t)}{dx}dx \\ &= -VI\Big|_0^H + \int_0^H I(x,t)\frac{dV(x,t)}{dx}dx \\ &= -VI\Big|_0^H - \frac{d}{dt}\left(\frac{1}{2}\int_0^H lI(x,t)^2 dx\right),\end{aligned} \tag{29}$$

where the last line follows from repeating the top line's steps with the inductive energy.

For the familiar case of an isolated line with open- or short-circuit terminations, the total energy is stored internally along the line. The voltage and current are each described by harmonic Fourier series, which are related through the telegrapher's equations. From orthogonality, each term in these series contributes independently to the total energy, and energy conservation pairs each voltage term with its corresponding current term linked by the telegrapher's equations. These pairs are known as the normal modes of the line (or standing-wave resonances), and the energy stored by each mode oscillates back and forth between being stored by the voltage term or by the current term.

However, returning to the example of a line terminated by a capacitor, the total energy also includes a contribution from the voltage across that capacitor. If you consider a snapshot in time when the voltage is zero everywhere, then all of that energy is instantaneously stored by the current on the line alone. Each term in the orthogonal, nonharmonic Fourier series for the current contributes independently to this energy, and together these contributions sum to the total energy. Thus, each current term corresponds to a normal mode and has a paired voltage term. However, picking a second snapshot in time when the current is zero everywhere, all of the energy is stored both by the voltage along line and across the capacitor. If only one mode has energy, then the sum of the energy stored by that mode's voltage along the line and that mode's voltage across the capacitor during the second snapshot equals the energy stored by that mode's current during the first snapshot. This snapshot-to-snapshot equality holds for each mode, provided only one mode at a time has nonzero energy.

But, what if multiple modes have energy? Consider again the second snapshot in time with zero current. If there were two modes with energy, then the nonorthogonality of their voltage terms would add a new contribution to the energy. That is, the integral for $U(t)$ still gives not only the same contributions present for each mode when they are alone, but also a new contribution from their product: for $V(x,t) = V_1(x,t) + V_2(x,t)$, $(c/2)\int_0^H V(x,t)^2 dx = (c/2)\int_0^H [V_1(x,t)^2 + V_2(x,t)^2 + 2V_1(x,t)V_2(x,t)]dx$, where the last piece is new. There is also a corresponding new contribution in the energy stored by the capacitor: for a voltage $V(H,t)$ across the capacitor, its energy is proportional to $[V_1(H,t)+V_2(H,t)]^2 = V_1(H,t)^2 + V_2(H,t)^2 + 2V_1(H,t)V_2(H,t)$, where the last piece is new. From considering the case of only one mode, we know energy is represented correctly when the new contributions from the line and from the capacitor are removed, so they must cancel. This picture scales to multiple modes on the line, because quadratic products in the line and termination capacitor energies may be rearranged as $\left(\sum_{n=1}^N a_n\right)^2 = \sum_{n=1}^N \left(a_n^2 + 2\sum_{m=n+1}^N a_n a_m\right)$.

Likewise, if the line was terminated by an inductor instead of a capacitor, then the same arguments suggest that the situation would reverse: the current series would be nonorthogonal, and the voltage series orthogonal, recovering in a particular case the orthogonal, nonharmonic sine series described above. For more complicated arrangements with a line connected to network(s) of both inductors and capacitors, at one of both of its ends and/or interior points, then most likely neither voltage or current would have orthogonal series along the line. Instead, orthogonality likely returns only if all parts of the entire system, network(s) and line, are included in the calculation of the total energy, or perhaps if alternate observables exist that correspond to all energy being stored on the line. Some modification of the above arguments is expected to accommodate losses in the line and its terminations.

Last, but not least, the same arguments suggest that for more complicated lossless lines with varying parameters along their length [$l \to l(x)$ and $c \to c(x)$], any orthogonality generalizes to follow the updated form for the energy stored by the line, $U(t) = \int_0^H [l(x)I(x,t)^2 + c(x)V(x,t)^2]\, dx/2$. For a worked example, see Ref. 5.

REFERENCES

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