Deriving the equivalent circuit of a Tesla coil

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https://bartmcguyer.com/notes/note-11-TcEquations.pdf

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TL;DR: A toy-model derivation of the equivalent circuit of a Tesla coil that modifies the approach of [PLoS ONE 9, e115397 (2014)] to more conveniently include a capacitive load. Approximates the secondary coil as a uniform transmission line leading to the "Miller" self-capacitance of a solenoid inductor [Proc. IRE 7, 299 (1919)].

CONTENTS

I. INTRODUCTION

This note supplements Reference 1, which is freely available online (open access). You will need a copy of Ref. 1 to follow the derivation below. To avoid confusion, the numbering of the equations and figures below continues after those in Ref. 1.

Briefly, Ref. 1 outlines how to generate an equivalent circuit (lumped-element model) for a transmission line coupled to one or more additional circuits. As a specific example, it derives an equivalent circuit for a Tesla coil (or Tesla transformer) by approximating its secondary coil as a uniform transmission line. For simplicity, it ignores the capacitive loading from an output electrode, which is normally significant in practice. While it later outlines how to include such a load, its general approach leads to a rather complex equivalent circuit. This note modifies that approach to more conveniently include a capacitive load, leading to the more familiar, simpler equivalent circuit for a Tesla coil.

I hope that the combination of this note with Ref. 1 will fill a pedagogical gap by semi-quantitatively justifying the ubiquitous equivalent circuit for a Tesla coil: Together, they provide a middle ground between introductory references that simply assert such a circuit and advanced references with field solutions for waves on ideal, infinite solenoids (e.g., sheath helix models). While much of the approach of this combination isn't new, it doesn't seem to have been revisited much since the early 20th century.[2] For example, the approach readily exposes potential issues with nonuniqueness and nonreciprocity in such circuits that seem to be mostly forgotten today.[1,3]

II. DERIVATION

Consider the physical setup sketched in Fig. 7(a) of a solenoid coupled with a lumped inductor that forms the air-core transformer in a Tesla coil. This is a modification of Fig. 1(a) in Ref. 1 that includes a lumped capacitive load C_{load} to model the effective electrostatic self-capacitance of an output electrode. Let's derive an equivalent circuit for this setup of the form shown in Fig. 7(b).

A. Modifying the Fourier series

Physically, adding the capacitive load C_{load} to the top of the solenoid alters its resonant modes. For example, if C_{load} is very small, then the modes should be very nearly the same as the quarter-wave resonances of an unloaded solenoid. However, if C_{load} is large enough that the top is effectively shorted, then we should expect half-wave resonances with a fundamental mode that serves as a "lumped-element" limit with uniform current throughout the solenoid. In between, the modes will depend on the exact value of C_{load}.

One way to handle this variation is to modify the spatial Fourier series so that its terms always correspond to resonant modes. We can do this by slightly modifying (3a,b) to be

$$V(x,t) = \sum_{\nu=1}^{\infty} \sin(k'_\nu x) V_\nu(t) \tag{27}$$

$$I(x,t) = \sum_{\nu=1}^{\infty} \cos(k'_\nu x) I_\nu(t), \tag{28}$$

where prime have been added to the wavenumbers k'_ν to indicate that their values now depend parametrically on C_{load}. The allowed values of k'_ν will be determined by a boundary condition at $x = H$ in the next section. For $C_{\text{load}} = 0$, the wavenumbers will be the same as in Ref. 1 for the unloaded case.

To use this modified series, we will have to revisit the approach of Ref. 1, because this new series has different properties than the original.[4] Specifically, only the cosine terms are guaranteed to form an orthogonal basis, and that basis is an unusual, nonharmonic Fourier series. The sine series is in general not orthogonal, and both series have term-dependent normalization that depends on C_{load}. We'll return to this in the next sections.

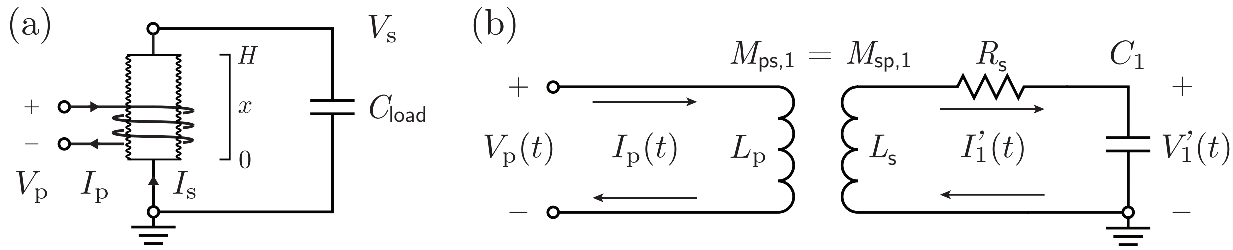

FIG. 7. Problem setup. (a) Physical setup of primary and secondary inductors with low-frequency self-inductances L_p and L_s. This is a modified version of Fig. 1(a) in Ref. 1 that includes a lumped capacitive load C_{load}. (b) Equivalent circuit to be derived for the setup at frequencies near the first self-resonance, which has the conventional form expected for a Tesla coil.

Nevertheless, the general approach outlined in Ref. 1 still applies. Each ν-th pair of terms generates an equivalent circuit, similar to Fig. 3(a), that corresponds to a resonant mode of the solenoid with C_{load}. These mode equivalent circuits stitch together to form a full equivalent circuit, similar to Fig. 5, through their coupling with the primary coil.

For a Tesla coil, note that we're really only interested in the modified fundamental mode with $\nu = 1$ that corresponds to the lowest resonance of the solenoid with C_{load}. In this case, the measurable top voltage and base current in Fig. 7(a) are approximately

$$V_s(t) = V(H,t) \approx \sin(k_1' H) V_1(t) \quad (29)$$

$$I_s(t) = I(0,t) \approx I_1(t), \quad (30)$$

and the equivalent circuits for the higher modes ($\nu > 1$) can usually be neglected.

B. Boundary condition from a capacitive load

While the boundary condition of $V(0,t) = 0$ at the grounded base is unchanged, the boundary condition at the top is now

$$I(H,t) = C_{load} \frac{\partial}{\partial t} V(H,t). \quad (31)$$

This condition will be satisfied if for each mode

$$I_\nu(t) \cos(k_\nu' H) = C_{load} \frac{d}{dt} V_\nu(t) \sin(k_\nu' H). \quad (32)$$

Jumping ahead, we will find in the next section that Eq. (8b) still applies for the modified series. We are interested in the case with $g = 0$ and $i_{sp}(x,t) = 0$ that reduces (8b) to

$$I_\nu(t) = \left(\frac{c}{k_\nu'}\right) \frac{d}{dt} V_\nu(t), \quad (33)$$

which used (4)–(7) to substitute $\widetilde{C}_\nu = c/k_\nu \longrightarrow c/k_\nu'$.

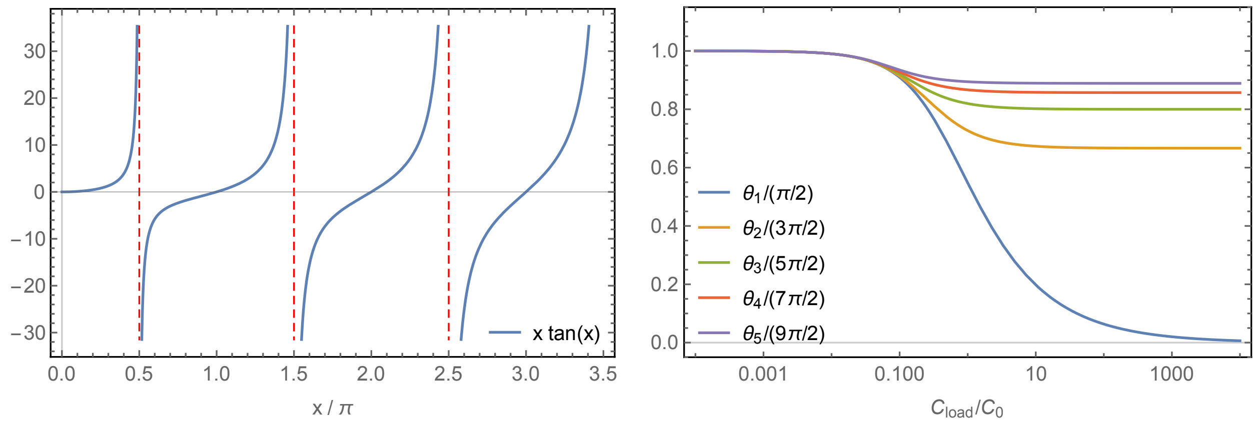

FIG. 8. Boundary condition and its solutions. (Left) Left hand side of the transcendental equation (36). Each solution corresponds to the x-axis value where a horizontal line would intersect each curve. (Right) First five solutions $x = \theta_\nu$ divided by their quarter-wave limits versus $1/y = C_{\rm load}/C_0$. The fractional change to reach the half-wave limit decreases as ν increases.

To proceed, let's introduce some handy notation. First, let's introduce the capacitance

$$C_0 = cH, \tag{34}$$

which physically is the quasistatic self capacitance of the solenoid with a uniform voltage profile. In practice, this is usually not exactly the same as the electrostatic (or low frequency) self-capacitance because c is frequency dependent, and though it's for the solenoid alone, it includes perturbations from the environment (e.g., electrode and primary coil). Next, let's introduce electrical full lengths of the solenoid for the modes,

$$\theta_\nu = k'_\nu H, \tag{35}$$

which are the solutions of the transcendental equation. Note that in Ref. 1 it was more convenient to work with the half lengths $\phi_\nu = k_\nu H/2$ because of a focus on reciprocity.

With these notation changes, combining (32) and (33) simplifies the boundary condition to the following transcendental equation that is shown in Fig. 8:

$$\boxed{\theta_\nu \, \tan(\theta_\nu) = \frac{C_0}{C_{\rm load}}} \tag{36}$$

This can be used to numerically solve for the mode lengths θ_ν and, equivalently, the modified wavenumbers k'_ν. Note that though deriving this involved dividing by $\cos(\theta_\nu)$, the case of $\cos(\theta_\nu) = 0$ is still captured by the limiting behavior of (36). Inspecting Fig. 8, the solutions fall within the ranges

$$\theta_\nu \in [(\nu-1)\pi, (2\nu-1)\pi/2] \le 2\phi_\nu \tag{37}$$

and the modified wavenumbers are always smaller than the original quarter-wave values,

$$k'_\nu \le k_\nu. \tag{38}$$

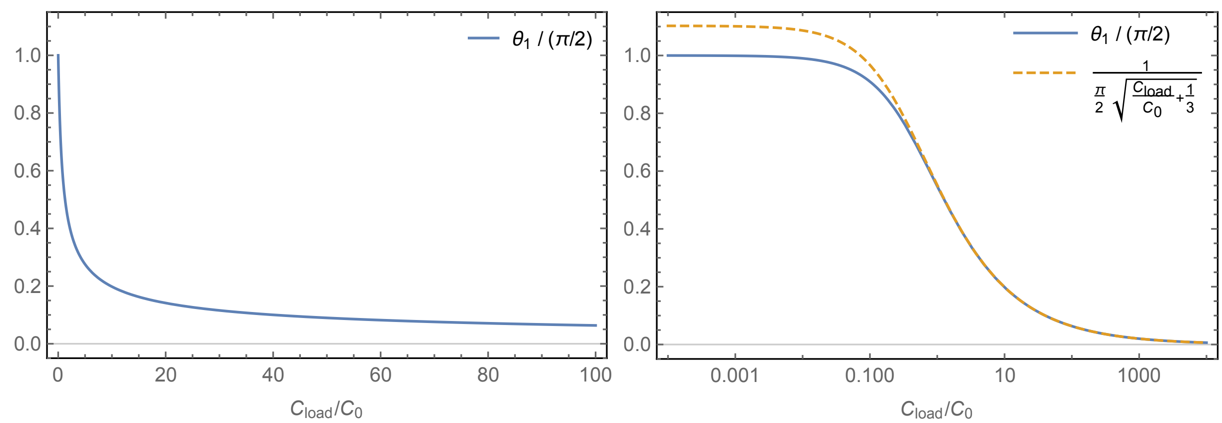

FIG. 9. Highlight of the fundamental solution θ_1 of the transcendental equation (36). Linear (left) and log-linear (right) plots. Note that all values (y-axes) are normalized to $\pi/2$. The dashed line is an approximation leading to the Miller self-capacitance for $C_{\text{load}} \gg C_0$.

In particular, the length θ_1 of fundamental solution is within the range

$$0 < \theta_1 \leq \pi/2 \tag{39}$$

and depends on C_{load}/C_0 as highlighted in Fig. 9.

C. Series orthogonality and normalization

Now that we have finished constructing the modified Fourier series (27) and (28), we can proceed to use them. To do so, we will need to make some slight adjustments. This will be covered in the next section, which updates the specific example in Ref. 1 to use the modified series. Before we continue, though, we should examine these series and discuss things a bit more. To help with this, consider the following integrals:

$$A(a,b) = \int_0^H \sin(ax)\sin(bx)dx = \begin{cases} 0 & a=b=0 \\ \frac{H}{2} - \frac{\sin(2aH)}{4a} & a=b\neq 0 \\ \frac{a\cos(aH)\sin(bH)-b\cos(bH)\sin(aH)}{b^2-a^2} & a\neq b, a \text{ or } b\neq 0 \end{cases} \tag{40}$$

$$B(a,b) = \int_0^H \cos(ax)\cos(bx)dx = \begin{cases} H & a=b=0 \\ \frac{H}{2} + \frac{\sin(2aH)}{4a} & a=b\neq 0 \\ \frac{a\sin(aH)\cos(bH)-b\sin(bH)\cos(aH)}{a^2-b^2} & a\neq b, a \text{ or } b\neq 0 \end{cases} \tag{41}$$

$$C(a,b) = \int_0^H \sin(ax)\cos(bx)dx = \begin{cases} 0 & a=b=0 \\ \frac{\sin(aH)^2}{2a} & a=b\neq 0 \\ \frac{a-a\cos(aH)\cos(bH)-b\sin(bH)\sin(aH)}{a^2-b^2} & a\neq b, a \text{ or } b\neq 0 \end{cases} \tag{42}$$

The original Fourier series in (3a,b) of Ref. 1 is a typical harmonic series. $A(k_n, k_m)$ shows that the set of sine terms is orthogonal, and $B(k_n, k_m)$ shows that the set of cosine terms is

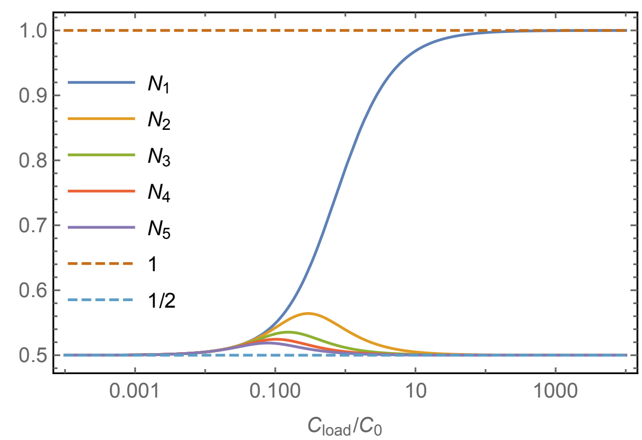

FIG. 10. Normalization length factors N_ν of (43) for the first five modes and their limiting values.

orthogonal. Perhaps atypical, $C(k_n, k_m)$ shows that the two sets are not mutually orthogonal, since they are each quarter-wave Fourier series instead of full-wave Fourier series over the domain $x \in [0, H]$. However, that doesn't affect their use in Ref. 1. The normalization for each term follows from $A(k_\nu, k_\nu) = B(k_\nu, k_\nu) = H/2$.

The modified series is unusual, as hinted above. $B(k'_n, k'_m)$ shows that the set of cosine terms in (28) is orthogonal for wavenumbers satisfying the boundary conditions (36). However, $A(k'_n, k'_m)$ shows that the set of sine series in (27) is no longer guaranteed to be orthogonal for those wavenumbers, except in the limits of null or infinite C_{load}. $C(k'_n, k'_m)$ shows that the two sets are not mutually orthogonal, as before. Normalization still follows from $A(k'_\nu, k'_\nu)$ and $B(k'_\nu, k'_\nu)$ but now varies for each term and depends on C_{load}. For convenience later, let's introduce the normalization length factors

$$N_\nu = B(k'_\nu, k'_\nu) = \frac{H}{2}\left[1 + \frac{\sin(2\theta_\nu)}{2\theta_\nu}\right] \geq 0. \tag{43}$$

Fig. 10 shows the first five N_ν. For the quarter-wave series in Ref. 1, $N_\nu = H/2$.

The set of cosine terms of the modified series form a nonharmonic Fourier series. For more information about this series, and to see example plots of its terms, please see Ref. 4. This set is a complete basis for current along the solenoid according to Sturm-Liouville theory. The set of sine terms can be adjusted to also be complete, but for different boundary conditions than considered here.[4]

The effect of increasing C_{load} can be interpreted as stretching the domain of the quarter-wave terms, or shrinking the solenoid, as sketched in Fig. 11. That the series is nonharmonic means that the effective, stretched lengths $H'_\nu = Hk_\nu/k'_\nu = (2\nu - 1)\pi/(2k'_\nu)$ equivalent to the original series domain vary for each term, except in the quarter-wave limit that gives the original series with the lengths aligned, $H'_\nu \longrightarrow H$. Similarly, in the half-wave limit the effective, stretched lengths $2H'_\nu$ become aligned, $2H'_\nu \longrightarrow H$, except for the special case of the fundamental mode which has an infinite domain ($H'_1 \longrightarrow \infty$ because $k'_1 \longrightarrow 0$).

Since we're really only interested in the fundamental term, it's tempting to force the series to be harmonic by aligning the higher-order lengths to be the same as that of the fundamental

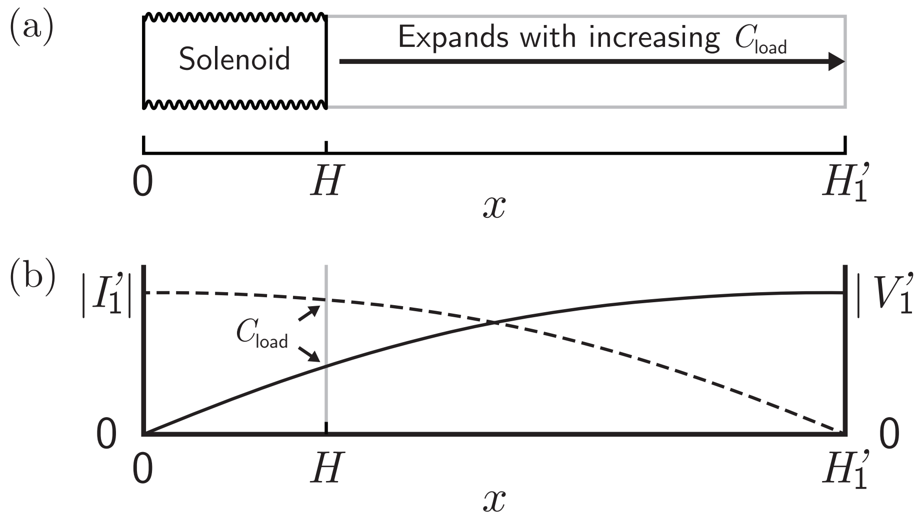

FIG. 11. Stretching the Fourier series into anharmonicity. (a) An increasing load alters the fundamental mode in a way that appears as if its domain is stretching, or alternatively, the solenoid is shrinking. This stretching (or shrinking) is not the same for all modes, producing a nonharmonic series. (b) Voltage and current spatial profiles for the fundamental Fourier series terms continued beyond the solenoid height to complete a quarter wavelength at H_1'. The boundary condition for C_{load} occurs at the actual top of the solenoid at $x = H$.

term, $H_\nu' = H_1'$. This would turn the modified series into a harmonic, quarter-wave Fourier series over a larger domain than the solenoid: $x \in [0, H_1']$. While the fundamental mode would correspond to an actual resonant mode, the higher modes would no longer do so. The sine and cosine series would both be complete, but would actually be "over complete" for the solenoid's shorter domain. As a result, the coupling with the primary coil would no longer be represented correctly without careful treatment. However, other circuit parameters for the fundamental mode would be the same, such as C_1. Therefore, to avoid such issues, let's proceed with the modified nonharmonic series.

Last but not least, a clarification about the "lumped-element" regime of $C_{\text{load}} \gg C_0$: Note that in the limit of $C_{\text{load}} \longrightarrow \infty$, there is effectively a short at the top $x = H$. In this limit, the modified series is a half-wavelength series with a voltage node at $x = H$ for each term in (27), so the series cannot represent a nonzero voltage exactly at the top. However, for finite C_{load} approaching this limit, the $\nu = 1$ terms are a nearly constant current and a nearly linear voltage increase across the solenoid. As C_{load} becomes infinite, the $\nu = 1$ voltage term eventually becomes zero across the solenoid. Therefore, the regime of $C_{\text{load}} \gg C_0$ for finite C_{load} does capture lumped-element behavior, but the actual limit of infinite C_{load} represents a solenoid grounded at both ends.

D. Specific example, revisited

To model a Tesla coil, let's update the specific example in Ref. 1 to use the modified Fourier series. Let's ignore shunt conductance ($g = 0$), which is negligible for a Tesla coil. Let's assume $i_{\rm sp}(x,t) = 0$ so that there is only distributed coupling to the primary coil following (15). Furthermore, let's keep the results general and wait to determine the final circuit parameters and circuit variables (voltages and currents) in the next section.

Then, using the modified series (27) and (28) with the Telegrapher's equations (2a,b), Eqs. (8a,b) become

$$V_\nu(t) = -\left(\widetilde{R}'_\nu + \widetilde{L}'_\nu \frac{d}{dt}\right) I_\nu(t) - \left(\frac{1}{k'_\nu N_\nu}\right) V'_{{\rm sp},\nu}(t) \tag{44}$$

$$I_\nu(t) = \widetilde{C}'_\nu \frac{d}{dt} V_\nu(t). \tag{45}$$

The first line follows from the orthogonality of the cosine series (28) with a corrected normalization using (41) and (43): Note that $k'_\nu N_\nu$ plays the role of ϕ_ν in (8a), and is equivalent to ϕ_ν for $C_{\rm load} = 0$. The source term $V'_{{\rm sp},\nu}(t)$ from the primary coil follows from (9) and has a prime to denote using k'_ν instead of k_ν. The second line satisfies (2b) if it holds for each mode. The primes on the circuit parameters denote their dependence on k'_ν instead of k_ν in (4)–(7). Note that (45) is equivalent to (33) asserted earlier.

To consider different circuit parameters, we can use (13) to introduce the circuit variables

$$V'_\nu(t) = \frac{n_\nu}{\alpha_\nu} V_\nu(t) \tag{46}$$

$$I'_\nu(t) = \frac{1}{n_\nu \alpha_\nu} I_\nu(t). \tag{47}$$

Here, the primes follow from the notation of Ref. 1. Substituting, Eqs. (44) and (45) become

$$V'_\nu(t) = -\left(R_\nu + L_\nu \frac{d}{dt}\right) I'_\nu(t) - M_{{\rm sp},\nu} \frac{d}{dt} I_{\rm p}(t) \tag{48}$$

$$I'_\nu(t) = C_\nu \frac{d}{dt} V'_\nu(t) \tag{49}$$

where the circuit parameters follow from (4)–(6), and are given by the lengths

$$A_\nu = R_\nu / r = L_\nu / l = n_\nu^2 / k'_\nu \tag{50}$$

$$B_\nu = C_\nu / c = 1/(n_\nu^2\, k'_\nu) \tag{51}$$

which used $\chi_\nu = n_\nu^2$ as noted after (13). The forward mutual inductance parameters $M_{{\rm sp},\nu}$ in (48) follow from (9) with (14), (15), and (24) and include coefficients from (46), giving

$$M_{{\rm sp},\nu} = \left(\frac{n_\nu}{\alpha_\nu k'_\nu N_\nu}\right) M'_\nu, \tag{52}$$

where M_ν is given by (24) and the prime indicates M'_ν uses k'_ν instead of k_ν.

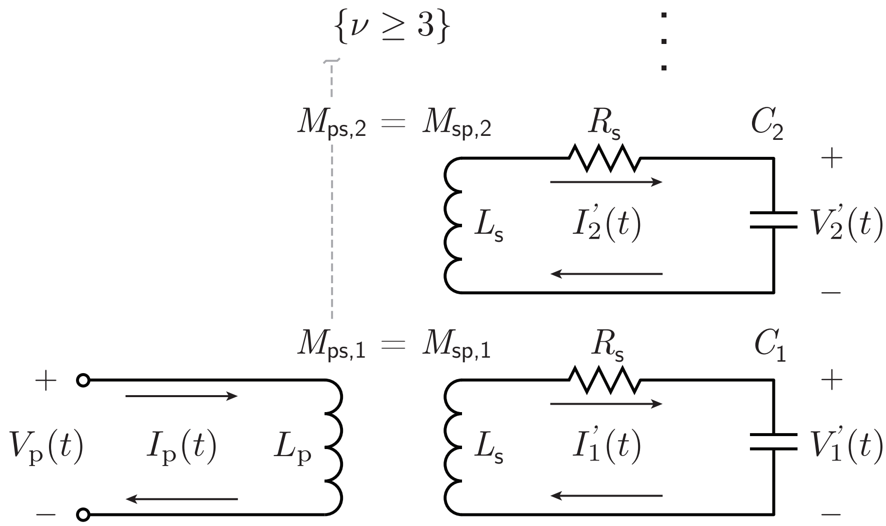

FIG. 12. Exact equivalent circuit given by (48), (49), and (53) for the setup in Fig. 7(a) using the modified Fourier series. The narrowband circuit in Fig. 7(b) is an approximation that ignores the contributions of the modes $\nu \geq 2$. This is a modified version of Fig. 5 in Ref. 1.

For the primary coil, from Fig. 3(b) the equivalent circuit is

$$V_\mathrm{p}(t) = -L_\mathrm{p}\frac{d}{dt}I_\mathrm{p}(t) - \sum_{\nu=1}^{\infty} M_{\mathrm{ps},\nu}\frac{d}{dt}I'_\nu(t) \tag{53}$$

with reverse mutual-inductance parameters

$$M_{\mathrm{ps},\nu} = \alpha_\nu n_\nu M'_\nu \tag{54}$$

that follow from (16) with (15), (18), (20)–(22), and (24) and include coefficients from (47).

That's it! Together, (48) and (49) define the mode equivalent circuits for the secondary coil and (53) defines the equivalent circuit for the primary coil. The circuit variables are given by (46) and (47). The circuit parameters are given by (50) and (51) and the mutual inductances by (52) and (54). Fig. 12 shows these equivalent circuits stitched together. For frequencies near the fundamental resonance, we can ignore all but the fundamental mode, and Fig. 12 reduces to Fig. 7(b). However, these circuits still have free parameters that we will choose in the next section.

E. Choosing equivalent-circuit parameters

To finish generating the equivalent circuit, we need to choose the circuit parameters. For each mode equivalent circuit, there are two sets of free parameters left to determine. To select them, let's follow the conventions in the typical equivalent circuit for a Tesla coil.

First are the parameters α_ν, which influences the mutual inductances and the circuit variables (voltages and currents). Following nearly universal convention,[1] let's choose for the mutual inductances to be reciprocal, $M_{\mathrm{sp},\nu} = M_{\mathrm{ps},\nu}$. From (52) and (54), this requires

$$\alpha_\nu = 1/\sqrt{k'_\nu N_\nu}, \tag{55}$$

which is a generalization of the condition $\alpha_\nu = 1/\sqrt{\phi_\nu}$ as stated after (13). In other words, we're choosing to avoid the curious artificial nonreciprocity studied in Ref. 1.

Last are the parameters $\chi_\nu = n_\nu^2$, which set the impedances modeled by the mode equivalent circuits. Again, following the convention for Tesla coils and the so-called classical model of an inductor, as discussed on p. 13 of Ref. 1, let's choose for the self inductance of the fundamental mode to be equal to the solenoid self inductance, $L_1 = L_\mathrm{s}$. For simplicity, let's chose this for all modes, $L_\nu = L_\mathrm{s} = lH$, which requires the lengths

$$A_\nu = H \longrightarrow \chi_\nu = n_\nu^2 = \theta_\nu. \tag{56}$$

As a result, this gives the remaining lengths

$$B_\nu = H/\theta_\nu^2. \tag{57}$$

With these choices, the equivalent circuit parameters are then

$$L_\nu = lH = L_\mathrm{s} \tag{58}$$
$$R_\nu = rH = R_\mathrm{s} \tag{59}$$
$$C_\nu = C_0/\theta_\nu^2 \tag{60}$$
$$M_{\mathrm{sp},\nu} = M_{\mathrm{ps},\nu} = \sqrt{\frac{H}{N_\nu}}\, M'_\nu. \tag{61}$$

By choice, the inductance and resistance are constant and set at their low-frequency values L_s and R_s. Note that there is a subtlety in that the distributed coefficients l and r (and c) are in general frequency dependent, but our approach approximates them as frequency independent. The capacitances and mutual inductances, however, depend on C_load as shown in Fig. 13. We will return to the capacitances and inductances in the next sections.

The circuit variables are then

$$V'_\nu(t) = \theta_\nu \sqrt{\frac{N_\nu}{H}}\, V_\nu(t) \tag{62}$$
$$I'_\nu(t) = \sqrt{\frac{N_\nu}{H}}\, I_\nu(t). \tag{63}$$

For a Tesla coil where the fundamental mode dominates, the observables are approximately

$$V_\mathrm{s}(t) \approx \sin(\theta_1) V_1(t) = \frac{\sin(\theta_1)}{\theta_1}\sqrt{\frac{H}{N_1}}\, V'_1(t) \tag{64}$$
$$I_\mathrm{s}(t) \approx I_1(t) = \sqrt{\frac{H}{N_1}}\, I'_1(t), \tag{65}$$

Fig. 13 shows how these variables and observables are related. The equivalent circuit in Fig. 7(b) misrepresents voltage and current except in the half-wave limit of large C_load.

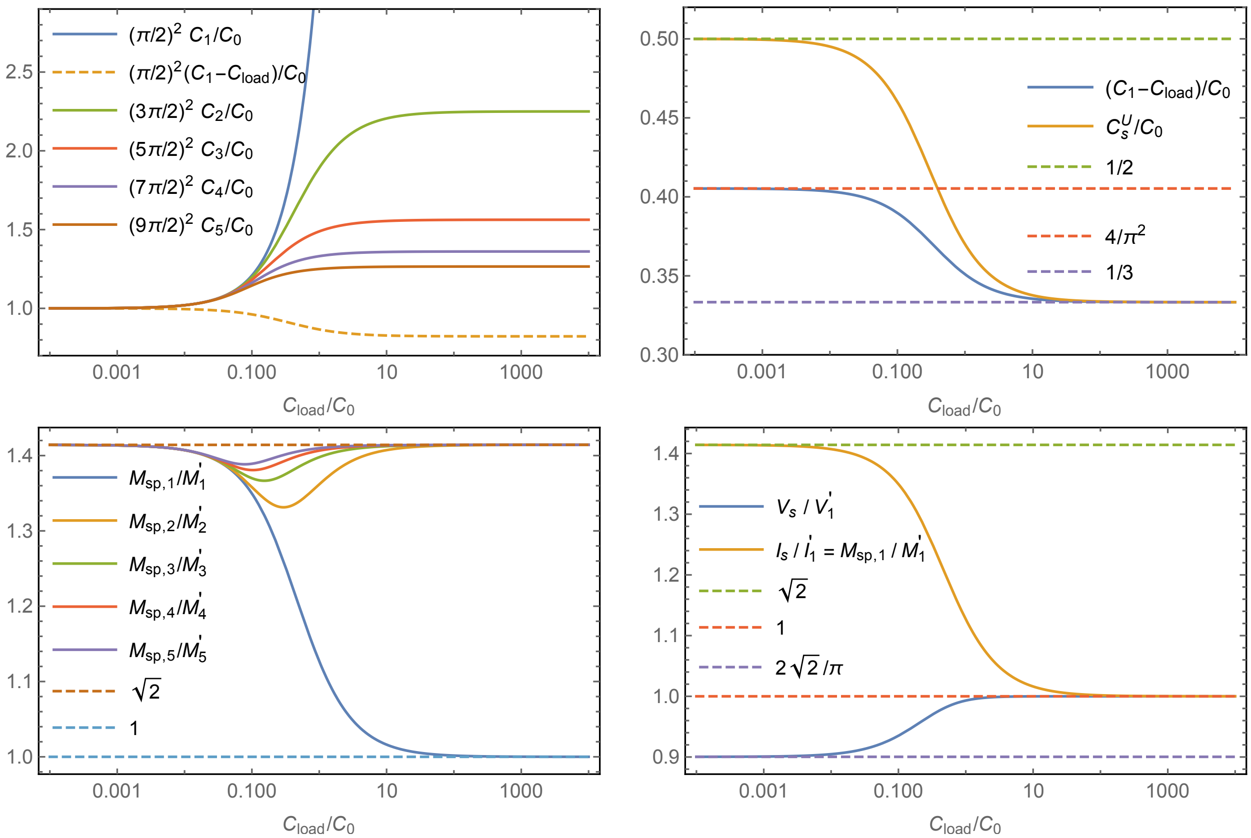

FIG. 13. Circuit parameters. (Top left) Capacitances C_ν of (60) normalized to C_0 for the quarter-wave limit. The dashed curve is a modification of C_1 to be finite in the half-wave limit. (Top right) Closeup of the behavior of C_1 and the energy-equivalent parameter $C_{\rm s}^U$ of (82). The 1/3 asymptote of the half-wave limit leads to the Miller capacitance (66). (Bottom left) Mutual inductances (61). (Bottom right) For the fundamental mode, circuit variables (62) and (63) of Fig. 7(b) versus the observable voltage (64) and current (65), which agree in the half-wave limit. Note that the asymptote of $V_{\rm s}/V_1' \longrightarrow 2\sqrt{2}/\pi \approx 0.9$ matches a case described on p. 14 of Ref. 1.

F. Miller self-capacitance of a solenoid inductor

In the regime of a large $C_{\rm load} \gg C_0$, the capacitance C_1 simplifies to the asymptotic value

$$C_1 \approx C_{\rm load} + \frac{1}{3} C_0 \tag{66}$$

as shown in Fig. 13(top right). This result was predicted at least as early as 1919 by J. M. Miller,[2] the famous electrical engineer after whom the "Miller capacitance" in amplifiers is named. Let's call the $C_0/3$ term in (66) the "Miller self-capacitance of a solenoid."

We can derive this analytically as follows. Before we do, note that there is no accepted analytical model for the empirical self-capacitance of single-layer solenoid inductors.[5] The Miller self-capacitance is an approximation that, as exactly written, neglects the energy stored by a voltage gradient along the solenoid. In the limiting case where inter-turn capacitances are negligible, it's a good approximation. Otherwise, you can show it's usually expected to be an underestimate. Several few years ago I tested this experimentally by wrapping a Tesla coil secondary with a variable thickness of paper and then an aluminum foil outer layer, to make an adjustable coaxial capacitor with the solenoid as the inner electrode[6]. This increased c enough that the inter-turn capacitance was negligible. Separate measurements of C_0 with an LCR meter and of C_1 from the lowest resonant frequency with a known C_{load} did indeed follow the relationship (66) even as C_0 was varied by adjusting the paper layer thickness. It also probed the similar result for weak loading with a coefficient of $4/\pi^2$ shown in Fig. 13(top right).

In the lumped regime, $\theta_1 \ll 1$ so the boundary condition (36) is approximately

$$x\tan(x) \approx x^2 + \frac{x^4}{3} + \frac{2x^6}{15} + \dots \tag{67}$$

Keeping only the first two terms gives

$$\theta_1^2 + \frac{\theta_1^4}{3} \approx \frac{C_0}{C_{\text{load}}}. \tag{68}$$

Dividing by θ_1^4 gives a quadratic equation for $1/\theta_1^2$. The positive solution to this equation is

$$\frac{1}{\theta_1^2} \approx \frac{C_{\text{load}}}{2C_0}\left(1 + \sqrt{1 + \frac{4C_0}{3C_{\text{load}}}}\right). \tag{69}$$

Approximating $\sqrt{1+x} \approx 1 + x/2$ simplifies the positive solution to

$$\frac{1}{\theta_1^2} \approx \frac{C_{\text{load}}}{C_0} + \frac{1}{3}. \tag{70}$$

This is a rather good approximation to θ_1 for $C_{\text{load}} \gtrsim C_0$ as shown in Fig. 9(b). Together with (60), this recovers (66).

1. Why is the coefficient 1/3?

The factor of 1/3 in (66) has a rather simple explanation. In the lumped regime near the half-wave limit, the circuit not only represents energy correctly but also represents voltage and current correctly. In this regime, the voltage $V(x,t) \approx (x/H)V_s(t)$. Therefore, the energy stored by capacitance should be

$$\frac{1}{2}\int_0^H c\left(\frac{x}{H}\right)^2 \langle V_s(t)^2\rangle dx = \frac{1}{2}\left(\frac{1}{3}C_0\right)\langle V_s(t)^2\rangle. \tag{71}$$

Pulling out just the part giving the factor of 1/3,

$$\frac{1}{H}\int_0^H \left(\frac{x}{H}\right)^2 dx = \frac{1}{3}. \tag{72}$$

Thus the origin of the coefficient 1/3 is that it's the normalized area of a parabola with unit height, which comes from the voltage distribution being approximately linear in the lumped regime for a solenoid grounded at one end and energy being the integral of voltage squared across the solenoid.

G. Energy-equivalent parameters

Separate from equivalent circuits, we can re-examine energy-equivalent parameters just as Ref. 1 did in (11) and (12). There are multiple ways to do this, making it a bit of a mess, so let's do most of them for clarity.

Using the modified series, the energy stored by each mode (11) becomes

$$U_\nu = \frac{1}{2}\int_0^H \left\langle l\, I_\nu(x,t)^2 + c\, V_\nu(x,t)^2 \right\rangle dx \tag{73}$$

$$= \frac{1}{2} L^U_\nu \left\langle I_\nu(t)^2 \right\rangle + \frac{1}{2} C^U_\nu \left\langle V_\nu(t)^2 \right\rangle + \frac{1}{2}\sum_{\mu\neq\nu} C^U_{\mu\nu} \left\langle V_\mu(t) V_\nu(t) \right\rangle \tag{74}$$

$$= \frac{1}{2} L'^U_\nu \left\langle I'_\nu(t)^2 \right\rangle + \frac{1}{2} C'^U_\nu \left\langle V'_\nu(t)^2 \right\rangle + \frac{1}{2}\sum_{\mu\neq\nu} C'^U_{\mu\nu} \left\langle V'_\mu(t) V'_\nu(t) \right\rangle . \tag{75}$$

The second line uses the circuit variables of the modified series and the third line uses the circuit variables of the equivalent circuits. There is a new term with mutual capacitances because in general the modified sine series (27) is not orthogonal. In the narrowband approximation where all but the fundamental mode can be ignored, the total energy is approximately

$$U_1 \approx \frac{1}{2} L^U_{\mathrm{s}} \left\langle I_{\mathrm{s}}(t)^2 \right\rangle + \frac{1}{2} C^U_{\mathrm{s}} \left\langle V_{\mathrm{s}}(t)^2 \right\rangle , \tag{76}$$

where the mutual capacitance terms are ignored.

The energy-equivalent inductance parameters are

$$L^U_\nu = l B(k'_\nu, k'_\nu) = l N_\nu = L_{\mathrm{s}} \frac{1}{2}\left[1 + \frac{\sin(2\theta_\nu)}{2\theta_\nu}\right] \tag{77}$$

$$L'^U_\nu = L^U_\nu \left(\frac{H}{N_\nu}\right) = L_{\mathrm{s}} = L_\nu \tag{78}$$

$$L^U_{\mathrm{s}} = L^U_1 . \tag{79}$$

The equivalence of the second line to the circuit parameter follows from choosing the equivalent circuits to be reciprocal and to have $L_\nu = L_{\mathrm{s}}$.

The energy-equivalent self capacitance parameters are

$$C^U_\nu = c A(k'_\nu, k'_\nu) = C_0 \frac{1}{2}\left[1 - \frac{\sin(2\theta_\nu)}{2\theta_\nu}\right] = \frac{1}{2}\left[C_0 - \sin(\theta_\nu)^2 C_{\mathrm{load}}\right] \tag{80}$$

$$C'^U_\nu = \frac{C^U_\nu}{\theta_\nu^2}\left(\frac{H}{N_\nu}\right) = C_\nu \left(\frac{2\theta_\nu - \sin(2\theta_\nu)}{2\theta_\nu + \sin(2\theta_\nu)}\right) = \frac{C_0}{\theta_\nu^2}\left(\frac{2\theta_\nu - \sin(2\theta_\nu)}{2\theta_\nu + \sin(2\theta_\nu)}\right) \tag{81}$$

$$C^U_{\mathrm{s}} = \frac{C^U_1}{\sin(\theta_1)^2} = \frac{1}{2}\left[\frac{C_0}{\sin(\theta_1)^2} - C_{\mathrm{load}}\right] . \tag{82}$$

Note that these only take in to account the energy stored by the solenoid, and do not include energy stored by C_{load}. The first line was simplified using the boundary condition.

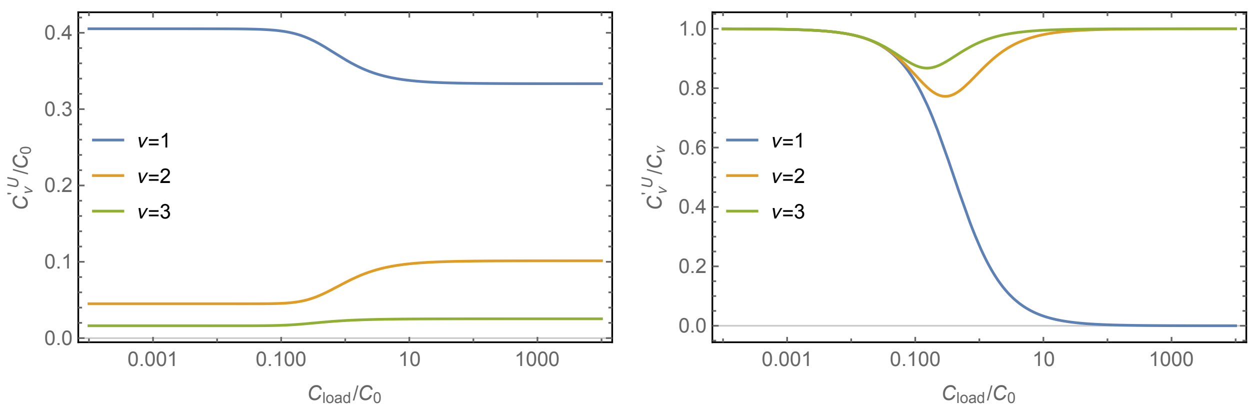

FIG. 14. Energy-equivalent capacitance parameters (81) plotted versus C_0 (left) or C_ν (right) for the first three modes.

The second line is interesting because its relationship to C_ν shows how C_ν takes in to account energy stored by $C_{\rm load}$. Parameters for the first three modes are plotted in Fig. 14. The last line is plotted in Fig. 13(top right) and recovers the Miller self-capacitance (66) in the lumped regime. To see how it does, note that $1/\sin(x)^2 \approx 1/x^2 + 1/3 + x^2/15 + \ldots$ for small x. Using (70) then gives $C^U_{\rm s} \approx C_0/3$.

The energy-equivalent mutual capacitance parameters are

$$C^U_{\mu\nu} = c\, C(k'_\mu, k'_\nu) \tag{83}$$

$$C'^U_{\mu\nu} = \frac{C^U_\nu H}{\theta_\mu \theta_\nu \sqrt{N_\mu N_\nu}}. \tag{84}$$

where the function $C(a, b)$ is given by (42).

III. DISCUSSION

This note derived the conventional equivalent circuit for a Tesla coil by modifying the approach of Ref. 1 to more conveniently include a capacitive load. The modified approach introduced a nonharmonic Fourier series to capture how the load alters the resonances of the secondary circuit. By construction, the nonreciprocity studied in Ref. 1 was avoided, but a related misrepresentation of voltage and current for a weak load is present. As asserted on p. 13 of Ref. 1, this approach reproduced the Miller[2] approximation of the self-capacitance of a single-layer solenoid inductor, which has no accepted analytical model.[5]

In contrast, the approach of Ref. 1 would've involved introducing a direct coupling at the top to introduce the capacitve load following the discussion after (15). This introduces an additional stitching mechanism that joins the equivalent circuits in Fig. 3(a) to produce a version of Fig. 5 that resembles Fig. 6(b). Many spatial modes would be needed to capture the circuit behavior with a non-zero load, so this would not directly lead to the conventional equivalent circuit for a Tesla coil.

Both of these approaches are limited by Ref. 1's approximation of a solenoid as a uniform transmission line, which neglects many effects from inter-turn capacitive and inductive couplings. The Telegraph equations can be modified to include those couplings, which would lead to some interesting corrections to this note. Such exploration has been done numerically for Tesla coils,[7] and predicts interesting phenomena like a current maximum inside the solenoid.

REFERENCES

[1]B. McGuyer, "Paul Drude's Prediction of Nonreciprocal Mutual Inductance for Tesla Transformers," PLoS ONE 9, e115397 (2014). Available online: `https://doi.org/10.1371/journal.pone.0115397`

[2]J. M. Miller, "Electrical oscillations in antennas and inductance coils," Proc. IRE 7, pp. 299–326 (1919). DOI: 10.1109/JRPROC.1919.217442

[3]B. H. McGuyer, "An old puzzle about reciprocity and Tesla coils," technical note, 2018. Available online: `https://bartmcguyer.com/notes/note-2-OldPuzzle.html`

[4]B. H. McGuyer, "Nonharmonic Fourier series," technical note, 2020. Available online: `https://bartmcguyer.com/notes/note-10-NonharmonicFourier.pdf`

[5]T. H. Lee, *Planar Microwave Engineering*, Cambridge, (2004). See the footnote in Chapter 6 on p. 142.

[6]B. H. McGuyer, "Test of the Miller self-capacitance of a solenoid inductor," technical note, 2021. Available online: `https://bartmcguyer.com/notes/note-12-MillerCapTest.pdf`

[7]Paul Nicholson and the Tesla Secondary Simulation Project Collaboration (2000). Theory Notes - Secondary Basics v1.0f. Available online: `http://abelian.org/tssp/pn2511.html`