Dipolar traveling-wave tube amplifiers

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TL;DR: A derivation of how traveling-wave tube amplifiers work that also explores what would happen if they used a beam of neutral particles with electric- or magnetic-dipole moments instead of electrons.

Traveling-wave tube amplifiers (TWTAs) use the interaction between an electron beam and a waveguide to amplify signals in the guide.[1–3] They're an enduring device from the fascinating history of vacuum tubes.[4] The classic example sends the beam down the center of a wire-helix guide (or single-layer solenoid), while others use cavities or delay lines. Importantly, the guide slows the signal so that it co-propagates with the beam at nearly the same velocity, enabling their interaction to transfer some of the beam's energy to the signal.

What would happen if you modified a TWTA to use a different beam or interaction? This note considers replacing the electrons with neutral particles that have permanent electric- or magnetic-dipole moments, such as certain atoms or molecules. It begins with an introductory treatment of a TWTA that connects its amplification with negative-resistance self-oscillation in RLC circuits. It then extends the treatment to electric- and magnetic-dipole interactions. A similar concept was explored before with magnetic bubbles in orthoferrite.[5]

CONTENTS

I. SETUP

Let's proceed similar to the traditional "Pierce" linear, small-signal analysis of a TWTA that considers a one-dimensional, non-relativistic beam of non-interacting particles coupled to an ideal transmission line as the waveguide. To begin, let's setup a shared treatment of the line, its dispersion relation, the particle beam, and the beam's evolution via the continuity equation. Then, in the following sections, let's finish each particular case by evaluating the particular forward (guide to beam) and backward (beam to guide) couplings. Together, we'll see that this closing of a loop—the forward coupling, beam evolution, and backward coupling—returns an ideal line but with modified parameters for each allowed propagation mode of the coupled beam-and-guide system. Solving the resulting dispersion relation determines these modes and their dynamics, that is, signal amplification, beam bunching, and beam slowing.

A. Waveguide

To model the waveguide, let's use an ideal $rlgc$ transmission line with voltage $V(x,t)$ and current $I(x,t)$ modeled by Telegraphers' equations,

$$\frac{\partial V}{\partial x} = -\left(r + l\frac{\partial}{\partial t}\right) I(x,t) + v_s(x,t) \tag{1}$$

$$\frac{\partial I}{\partial x} = -\left(g + c\frac{\partial}{\partial t}\right) V(x,t) + i_s(x,t), \tag{2}$$

where r, l, g, and c are distributed series resistance, series inductance, shunt conductance, and shunt capacitance. To model coupling with a beam, these equations include a distributed voltage source $v_s(x,t)$ and a distributed current source $i_s(x,t)$, which have units of voltage or current per length. By convention, positive current $I(x,t)$ flows towards increasing x (or along $\hat{x}$) in the guide.

Typically, the conductance is negligible ($g \approx 0$), so that a constant voltage offset V_0 has little effect. Likewise, adding a constant current offset I_0 (e.g., to provide a magnetic field to assist the beam) has little effect, apart from introducing a linear voltage gradient $\propto I_0 x$. Thus, let's ignore constant offsets going forward.

To proceed, let's consider harmonic traveling waves on the line of the form

$$V(x,t) = V_1 \exp(\gamma x - i\omega t) \qquad \text{and} \qquad I(x,t) = I_1 \exp(\gamma x - i\omega t), \tag{3}$$

with angular frequency ω and a so-called propagation constant $\gamma = \alpha + i\beta$ that has real-valued attenuation α and phase β (or wave number k) parts. These waves are the natural solution if we ignore sources and offsets, but we'll also recover them after treating the sources below. Before we continue, note that the above expressions are typical complex-valued phasors, which we can use because the Telegraphers' equations are linear. However, it'll be important below to remember that the actual distributions are the real parts of these and other complex functions.

B. Dispersion relation

While examining each case below, we'll find that the source terms can be rewritten as distributed impedance and admittance relations of the forms

$$v_s(x,t) = z_{\{v_s\}}(\omega,\gamma)I(x,t) \qquad \text{and} \qquad i_s(x,t) = y_{\{i_s\}}(\omega,\gamma)V(x,t). \tag{4}$$

Using these and the phasor forms (3) with the Telegraphers' equations gives a dispersion relation for the propagation constant:

$$\gamma^2 = \left(r - i\omega l - z_{\{v_s\}}\right)\left(g - i\omega c - y_{\{i_s\}}\right) = (r' - i\omega l')(g' - i\omega c'). \tag{5}$$

The last expression shows that the net effect of the source terms is to effectively modify the $rlgc$ line parameters for each solution of γ, by substituting $r \longrightarrow r' = r + \delta r$, $l \longrightarrow l' = l + \delta l$, $g \longrightarrow g' = g + \delta g$, and $c \longrightarrow c' = c + \delta c$, where $\delta g - i\omega\,\delta c = -z_{\{v_s\}}$ and $\delta r - i\omega\,\delta l = -y_{\{i_s\}}$.

As a result, despite having source terms, the line still carries harmonic traveling waves. The change is that the line now only carries these waves for each allowed solution of γ. These allowed waves are independent and superpose linearly, given our many approximations to come. They represent the coupled modes of the line-and-beam system, capturing how the beam affects the traveling waves in the line.

These waves have a propagation constant $\gamma = \pm\sqrt{(r' - i\omega l')(g' - i\omega c')}$, where the choice of sign determines the wave direction. Again using the Telegraphers' equations, their characteristic impedance $Z_0 = V_1/I_1 = (i\omega l' - r')/\gamma = \gamma/(i\omega c' - g') = \sqrt{(i\omega l' - r')/(i\omega c' - g')} \approx \sqrt{l'/c'}$. For weak total attenuation ($|r'/l' + g'/c'| \ll 1$), $(r' - i\omega l')(g' - i\omega c') \approx -\omega^2 l'c'[1 + i(r'/l' + g'/c')/\omega]$, so the real and imaginary parts of γ are $\alpha \approx -(r'/l' + g'/c')\beta/(2\omega)$ and $\beta \approx \pm|\omega|\sqrt{l'c'}$. The phase velocity $\omega/\beta \approx \mathrm{sgn}(\omega\beta)/\sqrt{l'c'}$, so the waves travel towards $\mathrm{sgn}(\omega\beta)\hat{x}$. The waves then grow along this direction if $\mathrm{sgn}(\alpha\beta\omega) = -\mathrm{sgn}(r'/l' + g'/c') > 0$. In this way, signal amplification results from negative attenuation in the modified line, similar to negative resistance in an RLC circuit.

Going back to the dispersion relation, if we approximate a lossless line ($r \approx g \approx 0$), or equivalently, assume that the guide-beam interaction dominates attenuation (or gain), then

$$\gamma^2 \approx \left(i\omega l + z_{\{v_s\}}\right)\left(i\omega c + y_{\{i_s\}}\right) = -\omega^2 lc + i\omega\left(l y_{\{i_s\}} + c z_{\{v_s\}}\right) + y_{\{i_s\}} z_{\{v_s\}}. \tag{6}$$

In the cases to follow, there'll be only one source term at a time, simplifying this further.

C. Particle beam

Let's assume we have a nonrelativistic beam of identical particles of mass M within a uniform, rotationally symmetric cross section of area A propagating towards positive x. To model this beam, let's use a speed distribution $u(x,t)$ and particle number density $n(x,t)$.[6] Again, let's allow these functions to be complex, but note that the beam actually follows their real parts. For example, these give a beam particle flux $J(x,t) = A\,\mathrm{Re}[n(x,t)]\,\mathrm{Re}[u(x,t)]$ towards $+\hat{x}$.

To evaluate these equations, let's consider only a finite region about $x = 0$ within which the changes in speed and density are small compared to their initial values of

$$u_0 = u(0,t) \qquad \text{and} \qquad n_0 = n(0,t), \tag{7}$$

which are constant in time, real valued, and positive. Within this region, let's introduce two new functions, $u_1(x,t)$ and $n_1(x,t)$, to model these small changes:

$$u(x,t) = u_0 + u_1(x,t) \qquad \text{with } |u_1(x,t)| \ll |u_0|, \text{ and} \tag{8}$$

$$n(x,t) = n_0 + n_1(x,t) \qquad \text{with } |n_1(x,t)| \ll |n_0|. \tag{9}$$

Note that these functions are zero at least at the initial position (and also for the pre-interaction region, $x \leq 0$): $u_1(0,t) = n_1(0,t) = 0$. To proceed below, let's evaluate everything to first order in these new functions. For example, the flux $J(x,t) \approx A(u_0 \operatorname{Re}[n_1(x,t)] + n_0 \operatorname{Re}[u_1(x,t)])$.

Note that the total derivative for beam quantities is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \left(\frac{\partial x}{\partial t}\right)\frac{\partial}{\partial x} = \frac{\partial}{\partial t} + u(x,t)\frac{\partial}{\partial x}, \tag{10}$$

because of the chain rule, which is important enough in fields like continuum mechanics to go by many different names (e.g., advective or material derivative).[a] This total derivative is absolutely critical to modeling the beam since it's composed of moving particles.[6] To evaluate forward couplings, we'll need the total derivative of the speed, which is

$$\frac{du}{dt} = \frac{\partial u_1}{\partial t} + u(x,t)\frac{\partial u_1}{\partial x} \approx \frac{\partial u_1}{\partial t} + u_0\frac{\partial u_1}{\partial x}, \tag{11}$$

where the final step dropped a term that was second order in u_1.

D. Continuity equation

For real-valued functions, particle conservation gives a local continuity equation (or transport equation) for the beam evolution:

$$\frac{\partial n}{\partial t} + \frac{\partial (nu)}{\partial x} = 0 \qquad \text{(assumes real-valued functions).} \tag{12}$$

However, this equation is nonlinear, so to accept complex-valued functions, it must be replaced with one that uses the real parts of the speed and density, such as

$$\frac{\partial}{\partial t}\frac{(n+\bar{n})}{2} + \frac{\partial}{\partial x}\frac{(n+\bar{n})(u+\bar{u})}{4} = 0 \qquad \text{(allows complex-valued functions),} \tag{13}$$

where a bar denotes complex conjugation. Evaluating either to first order gives

$$\frac{\partial n_1}{\partial t} + u_0\frac{\partial n_1}{\partial x} + n_0\frac{\partial u_1}{\partial x} \approx 0 \qquad \text{(first-order approximation of both).} \tag{14}$$

[a] https://en.wikipedia.org/wiki/Material_derivative

This equation will let us close a feedback loop between the forward and backward couplings treated below, because it gives a first-order equation for n_1:

$$\frac{\partial n_1}{\partial t} + u_0 \frac{\partial n_1}{\partial x} \approx -n_0 \frac{\partial u_1}{\partial x} \approx -n_0 \gamma\, u_1(x,t). \tag{15}$$

Here, anticipating upcoming results, the last step assumes a similar phasor form as (3). This differential equation evolves the beam, and we'll encounter similar equations for u_1 below. All of them share the form $\frac{\partial f}{\partial t} + A\frac{\partial f}{\partial x} = Be^{Cx+Dt}$, which has a general solution of $f(x,t) = Be^{Cx+Dt}/(D+AC)+E$ provided $D+AC \neq 0$, where capital letters are real-valued coefficients. Thus, the solution here is

$$n_1(x,t) \approx -\left(\frac{n_0\gamma}{u_0\gamma - i\omega}\right) u_1(x,t), \tag{16}$$

which assumes either $\alpha = \mathrm{Re}[\gamma] \neq 0$ (the type of solution of interest) or $\beta = \mathrm{Im}[\gamma] \neq \omega/u_0$ (phase velocity mismatch) to keep the denominator finite.

II. CLASSIC ELECTRIC-MONOPOLE TWTA (E0 INTERACTION)

In this section, let's finish the derivation of the classic TWTA for a beam of electrons.[2–4] The following sections then extend this analysis to neutral particles with dipole moments. In each case, the task is to evaluate the forward and backward couplings. Together with the continuity equation, as treated above, this leads to a dispersion relation to solve for the allowed values of γ.

A classic TWTA uses the electric-monopole (E0) interaction between the guide and an electron beam. The forward coupling is the Lorentz force of each electron's charge Q with the guide's electric field $\mathbf{E}$. The beam is placed so that this electric field is very nearly axial, such as near the center of a helix guide, and this forward coupling gives

$$M\frac{du}{dt} \approx Q\hat{x}\cdot\mathbf{E}(x,t) \approx QE_x(x,t) \approx -Q\frac{\partial V}{\partial x} \approx -Q\gamma\, V(x,t), \tag{17}$$

where the electron charge $Q < 0$. The final step assumed the form of (3), and assumed a negligible contribution from any other source of voltage gradients, such as a current offset I_0. Using (11), this gives an approximate, first-order differential equation for $u_1(x,t)$:

$$\frac{\partial u_1(x,t)}{\partial t} + u_0\frac{\partial u_1(x,t)}{\partial x} \approx -\left(\frac{\gamma Q}{M}\right) V(x,t). \tag{18}$$

This has the same form as the differential equation in (15) for $n_1(x,t)$, and its solution is

$$u_1(x,t) \approx -\left(\frac{\gamma Q/M}{u_0\gamma - i\omega}\right)[V(x,t) - V(0,t)], \tag{19}$$

which again requires either $\alpha = \mathrm{Re}[\gamma] \neq 0$ or $\beta = \mathrm{Im}[\gamma] \neq \omega$ to keep the denominator finite.

The backward coupling comes from the induced image charge from the beam. Noting that the beam current is $I_b(x,t) = QJ(x,t) = Q\,\mathrm{Re}[n]\,\mathrm{Re}[u]$, this coupling is approximately

$$\mathrm{Re}[i_s(x,t)] \approx -\frac{\partial I_b}{\partial x} = -AQ\frac{\partial\,\mathrm{Re}[n]\,\mathrm{Re}[u]}{\partial x} = AQ\frac{\partial\,\mathrm{Re}[n]}{\partial t}, \tag{20}$$

where the last step comes from the continuity equation (12). This is a special case of the Shockley-Ramo theorem approximating complete, local induction. Switching to phasors and using the solution (16), the backward coupling is an admittance relation,

$$i_s(x,t) = y_{\{i_s\}}(\omega,\gamma)V(x,t) \approx -\left(\frac{\omega n_0 AQ^2}{M}\right)\left(\frac{i\gamma^2}{(u_0\gamma - i\omega)^2}\right)V(x,t), \tag{21}$$

where the parenthesis separate real and complex coefficients.

A. Pierce dispersion relations and solutions

Using this with (6) gives a dispersion relation for a lossless line:

$$\gamma^2 \approx -\omega^2 lc - i\omega l\left(\frac{n_0 AQ^2\omega}{M}\right)\left(\frac{i\gamma^2}{(u_0\gamma - i\omega)^2}\right) = -\omega^2 lc + \left(\frac{l n_0 AQ^2\omega^2}{M}\right)\left(\frac{\gamma}{u_0\gamma - i\omega}\right)^2. \tag{22}$$

Interestingly, that sign of Q doesn't matter. Introducing an uncoupled propagation constant for the line,

$$\gamma_g = i\beta_g = i\omega\sqrt{lc} = i\omega/u_g\gamma_g, \tag{23}$$

and an uncoupled propagation constant for the beam,

$$\gamma_b = i\beta_b = i\omega/u_0, \tag{24}$$

then (22) is equivalent to the so-called Pierce 4-wave dispersion relation[2–4]

$$\left(\gamma^2 - \gamma_g^2\right)\left(\gamma - \gamma_b\right)^2 + 2\gamma_b\gamma_g\gamma^2 C^3 \approx 0, \tag{25}$$

where the dimensionless Pierce gain parameter C is given by

$$C^3 \equiv \frac{Z_0}{4|V_b/I_b(0,t)|} \approx -\left(\frac{l n_0 AQ^2\omega^2}{2Mu_0^2\gamma_g\gamma_b}\right) \geq 0. \tag{26}$$

The above approximate equality comes from using the characteristic impedance $Z_0 \approx \sqrt{l/c}$, the initial beam current $|I_b(0,t)| = |Q|J(0,t) = |Q|An_0u_0 > 0$, and the initial beam voltage $|V_b| = Mu_0^2/(2|Q|) > 0$. Rough values for C in practice are between 0.01 and 0.10.[4] (For example, $I_b(0,t) \approx 10$ mA, $V_b \approx 1.5$ kV, and $Z_0 \approx 150$ Ohms gives $C \approx 0.063$.) This Pierce dispersion relation ignores space-charge effects,[2–4] which are important for electron beams, and is often given in terms of β instead of γ [in which case β is likely modified to be complex valued: $\beta \longrightarrow \beta = i(\alpha - \gamma)$].[2,4]

Before we consider solutions, to help find "C" parameters for new dispersion relations below, let's rewrite (25) as a dimensionless quantity

$$\frac{\left(\gamma^2-\gamma_g^2\right)\left(\gamma-\gamma_b\right)^2}{\gamma_b\gamma_g\gamma^2} \approx -2C^3 = \left(\frac{Z_0}{2}\right)\left|\frac{I_b(0,t)}{V_b}\right| = \left(\frac{\omega^2 l}{2\gamma_b\gamma_g u_0}\right) Q^2 \left(\frac{J(0,t)}{M u_0^2/2}\right). \tag{27}$$

In the final expression, the first parenthesis gives the coupling details, then there's the interaction parameter (here, Q), and finally the beam dynamics (particle flux divided by the kinetic energy per particle). The coupling details reduce via $Z_0 = \omega^2 l/(\gamma_g\gamma_b u_0)$.

The Pierce dispersion relation (25) is a quartic polynomial with four solutions for γ. It's well studied, so here's how to get its solutions: Since C is a small parameter, let's look for the lowest-order solutions in C. To begin, if we set $C \longrightarrow 0$, which represents no beam-guide coupling, then (25) reduces to $(\gamma^2-\gamma_b^2)\left(\gamma-\gamma_g\right)^2 \approx 0$. By inspection, this has four solutions: a pair of "forward" and "backward" solutions for the line, $\gamma \approx \pm\gamma_g$, and a duplicate pair of forward solutions for the beam, $\gamma \approx \gamma_b$. This makes sense, since the lossless line carries the usual forward and backward solutions to the one-dimensional wave equation, and the non-interacting beam can only carry forward solutions up to its two degrees of initial condition freedom. Reintroducing a nonzero C, the typical approach is to look for synchronous (same speed) solutions by substituting $\gamma_b \approx \gamma_g$ in (25), which are the main ones of interest to TWTAs. Next, following a trick in Pierce's book,[3] we can find the three forward solutions by substituting $\gamma = \gamma_g(1 + C\delta)$ in (25) and expanding, which gives

$$2 + \delta(2 + C\delta)(2C + \delta^2) \approx 2(1 + \delta^3) + C\delta(4 + \delta^3) + \ldots \approx 2(1 + \delta^3) \approx 0. \tag{28}$$

The solution to $\delta^3 = -1$ are $\delta \in \{-1, (1 + i\sqrt{3})/2, (1 - i\sqrt{3})/2\}$. Thus, the three forward solutions are $\gamma \in \{(1 - C)\gamma_g, (1 + C/2)\gamma_g + (iC\sqrt{3}/2)\gamma_g, (1 + C/2)\gamma_g - (iC\sqrt{3}/2)\gamma_g\}$. In order, there's a constant, a decaying, and a growing (signal amplifying) forward wave. (As an aside, reducing the dispersion relation so that it only gives these three solutions gives a so-called Pierce 3-wave dispersion relation.[2]) Similarly, again borrowing from Pierce's book,[3] to get the backward solution, instead substitute $\gamma = \gamma_g(-1+C^3\delta)$ in (25) and expand, which gives $\delta \approx 1/4$ and thus $\gamma \approx (-1+C^3/4)\gamma_g$. This backwards wave is nearly untouched by the beam (C^3 is very small), which makes sense since it moves in an opposing direction from the beam. [Another aside: There is a related traveling-wave/velocity-modulating device called a backwards-wave oscillator (or BWO) that does use a counter-synchronous arrangement. Some O-type BWOs strongly resemble TWTAs, but they seem to differ by using a double helix or serpentine waveguide.] Finally, there are solutions for the non-synchronous case, including amplifying ones, but they're usually investigated numerically (see Appendix A).

Now that we've found a forward amplifying solution, let's return to signal amplification. With our approximations, the amplitude of phasors like (3) grows as it propagates a distance x by the multiplicative factor $G(x) = e^{|\alpha x|}$, using $\alpha = \mathrm{Re}[\gamma]$ of the amplifying solution. For a classic TWTA, $G(x) \approx e^{|C\gamma_g x\sqrt{3}/2|} \approx e^{|C\omega x\sqrt{3}/(2u_0)|}$. For a rough example, using $C \approx 0.05$, $u_0 \approx 0.5v_c$ (half the speed of light), $\omega \approx 2\pi \times 10$ GHz, and an interaction length of about 10 cm gives $G(10\text{ cm}) \approx 8$.

III. MAGNETIC-DIPOLE TWTA (M1 INTERACTION)

Suppose now that we replace the electrons with a beam of neutral particles ($Q \to 0$), and that each of these particles has a permanent (static) magnetic-dipole moment $\boldsymbol{\mu}$. Let's constrain this moment to always be along $\hat{x}$, so that $\boldsymbol{\mu} = \mu_1 \hat{x}$. (In practice, you could try applying a large axial magnetic bias field to maintain this alignment.) Let's assume μ_1 is constant (no torquing or state changes). To proceed, let's model each dipole as a current loop of radius $r_{\{\mu_1\}}$ and right-handed current $I_{\{\mu_1\}}$ so that $\mu_1 = \pi (r_{\{\mu_1\}})^2 I_{\{\mu_1\}}$. Fortunately, the exact values of $r_{\{\mu_1\}}$ and $I_{\{\mu_1\}}$ won't matter.

In this case, there's a magnetic-dipole (M1) interaction between the guide and the particle beam. The forward coupling is the force on each dipole from the guide's magnetic field $\mathbf{B}(x,t) \approx B_x(x,t)\hat{x}$. For simplicity, let's approximate this field using that of an infinite solenoid, $B_x(x,t) \approx \mu_0 n_h I(x,t)$, where n_h is the number of turns per length of the helix along $\hat{x}$, with a sign that accounts for the winding direction of the solenoid following the right-hand rule. This gives a force on the loop[7] of

$$M \frac{du}{dt} = \nabla (\boldsymbol{\mu} \cdot \mathbf{B}) \approx \mu_1 \frac{\partial B_x}{\partial x} \approx \mu_1 \mu_0 n_h \frac{\partial I}{\partial x} \approx \mu_1 \mu_0 n_h \gamma \, I(x,t). \tag{29}$$

Just as above, this leads to an equation for u_1 that we can solve, which gives

$$u_1(x,t) \approx \left(\frac{\mu_1 \mu_0 n_h \gamma / M}{u_0 \gamma - i\omega} \right) [I(x,t) - I(0,t)]. \tag{30}$$

Using this, the solution (16) to the continuity equation then gives n_1.

The backward coupling comes from variation in the magnetic flux that the beam induces in the guide. For our current loop model, this coupling occurs via a mutual inductance $\mathcal{M}$ between each loop and the solenoid. Taking advantage of reciprocity, let's calculate this inductance via the flux captured by each loop per unit current from the solenoid, which gives the magnitude $\mathcal{M} = \pi r_{\{\mu_1\}}^2 B_x(x,t)/I(x,t) = \pi r_{\{\mu_1\}}^2 \mu_0 n_h$. Finally, we need the total effective loop current that accounts for all particles at a given position x, which is $I_{\{\Sigma\mu_1\}}(x,t) = J(x,t) I_{\{\mu_1\}} = J(x,t)\mu_1 / \left(\pi r_{\{\mu_1\}}^2\right)$ as a real-valued quantity. Using these, the induced electromotive force (EMF) in the guide at x is $-\mathcal{M}\,(\partial/\partial t)\, I_{\{\Sigma\mu_1\}}(x,t)$. (Before we continue, let's double check this sign: Lenz's law says that the induced EMF should create an opposing magnetic flux. Here, if $I_{\{\Sigma\mu_1\}}(x,t)$ increases locally at x, then for positive n_h, we need $B_x(x,t)$ and thus $I(x,t)$ to decrease. Looking at the first of the Telegraphers' equations, this would occur if $v_s(x,t)$ decreases. Fingers crossed on this.) All together, the distributed voltage source is

$$\mathrm{Re}[v_s(x,t)] \approx -\mathcal{M} \frac{\partial^2 I_{\{\Sigma\mu_1\}}}{\partial x \partial t} = -\left(\frac{\mathcal{M}\mu_1 A}{\pi r_{\{\mu_1\}}^2} \right) \frac{\partial^2 \mathrm{Re}[n]\, \mathrm{Re}[u]}{\partial x \partial t} = \mu_1 \mu_0 n_h A \frac{\partial^2 \mathrm{Re}[n]}{\partial t^2}. \tag{31}$$

Taking care while converting to a phasor gives the impedance relation

$$v_s(x,t) = z_{\{v_s\}}(\omega,\gamma) I(x,t) \approx \mu_1 \mu_0 n_h A \frac{\partial^2 n_1}{\partial t^2} = \left(\frac{(\omega \mu_1 \mu_0 n_h)^2 n_0 A}{M} \right) \left(\frac{\gamma}{u_0 \gamma - i\omega} \right)^2 I(x,t). \tag{32}$$

Interestingly, the sign of μ_1 doesn't matter, just like for Q above.

A. Dispersion relation and solutions

Using this with (6) gives a dispersion relation for a lossless line:

$$\gamma^2 \approx -\omega^2 lc + i\omega c \left(\frac{(\omega\mu_1\mu_0 n_h)^2 n_0 A}{M} \right) \left(\frac{\gamma}{u_0\gamma - i\omega} \right)^2 . \tag{33}$$

Amazingly, this has the same form as the classic TWTA relation of (25),

$$\left(\gamma^2 - \gamma_g^2\right) (\gamma - \gamma_b)^2 + 2\gamma_b\gamma_g\gamma^2 C_{\mathrm{M1}}^3 \approx 0, \tag{34}$$

just with a new "C" parameter. Here, and subsequently, let's use the subscripts "M1" and "E0" to differentiate between parameters for the magnetic-dipole and classic cases. Rewriting this as a dimensionless quantity gives

$$\frac{\left(\gamma^2 - \gamma_g^2\right) (\gamma - \gamma_b)^2}{\gamma_b\gamma_g\gamma^2} \approx -2C_{\mathrm{M1}}^3 = \frac{i\omega c n_0 A (\mu_1\mu_0 n_h \omega)^2}{M u_0^2 \gamma_b \gamma_g} = \left(\frac{i\omega c (\mu_0 n_h \omega)^2}{2\gamma_b\gamma_g u_0} \right) \mu_1^2 \left(\frac{J(0,t)}{M u_0^2/2} \right) . \tag{35}$$

Using $i\omega c = \gamma_g/Z_0$ and $\omega^2 = -\gamma_b u_0$, the first coefficient simplifies to

$$\frac{i\omega c (\mu_0 n_h \omega)^2}{2\gamma_b\gamma_g u_0} = -\frac{(\mu_0 n_h)^2}{2Z_0}, \tag{36}$$

which gives an M1 gain parameter of

$$C_{\mathrm{M1}}^3 = \left(\frac{(\mu_0 n_h)^2}{4Z_0} \right) \mu_1^2 \left(\frac{J(0,t)}{M u_0^2/2} \right) \geq 0. \tag{37}$$

(Hopefully I didn't goof the sign. Appendix A provides some support.) Amazingly, this parameter is the same as the classic TWTA parameter,

$$C_{\mathrm{M1}} = C_{\mathrm{E0}}(Q \longrightarrow Q_{\{\mu_1\}} = \mu_1\mu_0 n_h/Z_0), \tag{38}$$

if the electron charge is substituted with an effective magnetic-dipole charge as shown. This allows an artificial comparison of this M1 case with the E0 case (see below).

For a quick order-of-magnitude estimate of the new Pierce gain parameter C_{M1}, let's consider a thermal beam of something like cesium atoms. Rough values are $\mu_1 \approx \mu_B$ (the Bohr magneton), $M \approx 133$ amu, $u_0 \approx 245$ m/s, $J(0,t) \approx 10^{10}$/s, $n_h \approx 1$/mm, and $Z_0 \approx 150$ Ohms. Together this gives $C_{\mathrm{M1}}^3 \approx -3.4 \times 10^{-25}$ and thus $C_{\mathrm{M1}} \approx -7 \times 10^{-9}$, which is about seven orders of magnitude smaller than the classic TWTA case. In terms of effective classic TWTA parameters, noting $\mu_0\mu_B/(|Q_e|\ \mathrm{Ohm}) \approx 7.3 \times 10^{-8}$, where Q_e is the electron charge, gives $Q_{\{M1\}}/|Q_e| \approx 10^{-9}$, which in turn gives an effective electron-beam voltage and current of 85 MV and 10^{-18} A. In short, the equivalent electron-beam impedance is huge, drastically shrinking the Pierce gain parameter.

This case also has the same form of solutions as the classic TWTA, so signal amplification has the same form: $G_{\mathrm{M1}}(x) \approx e^{|C_{\mathrm{M1}}\omega x\sqrt{3}/(2u_0)|}$. For a rough example, using $C_{\mathrm{M1}} \approx -7 \times 10^{-9}$, $u_0 \approx 245$ m/s (or about 0.8 ppm of the speed of light), $\omega \approx 2\pi \times 10$ GHz, and an interaction length of about 1 m gives $G_{\mathrm{M1}}(1\ \mathrm{m}) \approx 6$. Remarkably, we see that the severe reduction in the Pierce gain parameter is counteracted by the drastic reduction in speed. Note also that nothing about these parameters have been optimized, so finding any gain at all is fascinating.

IV. ELECTRIC-DIPOLE TWTA (E1 INTERACTION)

Now suppose that the neutral particles each have a permanent (static) electric-dipole moment $\mathbf{p}$, instead of a magnetic moment ($\mu_1 \to 0$). To simplify things, let's again constrain this moment to always be along $\hat{x}$, so that $\mathbf{p} = p_1\hat{x}$ with p_1 constant. (I'm not sure this is practical, because this alignment is likely to change.) To proceed, let's model each dipole as coming from a pair of displaced charges: a charge $+q_1$ advanced in position by $+d_1/2$ along $\hat{x}$, and a charge $-q_1$ delayed by $-d_1/2$. This gives $p_1 = q_1 d_1$, following typical convention. Fortunately, the exact values of q_1 and d_1 won't matter.

In this case, there's an electric-dipole (E1) interaction between the guide and the particle beam. The forward coupling is the force on each dipole from the guide's electric field $\mathbf{E}(x,t)$. Picking up where we left off with the classic TWTA, this is

$$M\frac{du}{dt} = (\mathbf{p}\cdot\nabla)\mathbf{E} \approx p_1\frac{\partial E_x}{\partial x} \approx -p_1\frac{\partial^2 V}{\partial x^2} = -p_1\gamma^2 V(x,t). \tag{39}$$

As above, this leads to an equation for u_1 that we can solve, which gives

$$u_1(x,t) \approx -\left(\frac{p_1\gamma^2/M}{u_0\gamma - i\omega}\right)[V(x,t) - V(0,t)]. \tag{40}$$

Using this, the solution (16) to the continuity equation then gives n_1.

The backward coupling still comes from the induced image charge of the beam, just as in the classic case. However, there are now two charges for each particle using our model. Let's consider the case of a vanishing separation d_1. Then, adapting (21) gives

$$\mathrm{Re}[i_s(x,t)] \approx \lim_{d_1\to 0} Aq_1\frac{\partial}{\partial t}\left[\mathrm{Re}[n(x+d_1/2,t)] - \mathrm{Re}[n(x-d_1/2,t)]\right] = Ap_1\frac{\partial^2\,\mathrm{Re}[n]}{\partial x\partial t} \tag{41}$$

Here, the limit became a derivative since $q_1 = p_1/d_1$. This gives the admittance relation

$$i_s(x,t) = y_{\{i_s\}}(\omega,\gamma)V(x,t) \approx -\left(\frac{p_1^2 n_0\omega A}{M}\right)\left(\frac{i\gamma^4}{(u_0\gamma - i\omega)^2}\right)V(x,t). \tag{42}$$

Interestingly, the sign of p_1 doesn't matter, just like for Q and μ_1 above.

A. Dispersion relation and solutions

Using this with (6) gives a dispersion relation for a lossless line:

$$\gamma^2 \approx -\omega^2 lc - i\omega l\left(\frac{p_1^2 n_0\omega A}{M}\right)\left(\frac{i\gamma^4}{(u_0\gamma - i\omega)^2}\right) = -\omega^2 lc + \left(\frac{l(p_1\omega)^2 A n_0}{M}\right)\left(\frac{\gamma^2}{u_0\gamma - i\omega}\right)^2. \tag{43}$$

This has almost, but not quite the same form as the previous E0 and M1 cases. This leads to a modified TWTA dispersion relation of the form

$$\left(\gamma^2 - \gamma_g^2\right)(\gamma - \gamma_b)^2 + 2\gamma^4 C_{\mathrm{E1}}^3 \approx 0. \tag{44}$$

Here, and subsequently, let's use the subscript "E1" to differentiate the new "C" parameter from the previous E0 and M1 cases. Note that the coupling term in this new dispersion relation has a different power of γ (quartic, not quadratic). Amazingly, repeating the same process to find synchronous solutions recovers the same solutions as the classic case!

Rewriting this as a dimensionless quantity gives

$$\frac{\left(\gamma^2-\gamma_g^2\right)\left(\gamma-\gamma_b\right)^2}{\gamma^4} \approx -2C_{\mathrm{E1}}^3 = \frac{l(p_1\omega)^2 A n_0}{M u_0^2} = \left(\frac{l\omega^2}{2u_0}\right) p_1^2 \left(\frac{J(0,t)}{M u_0^2/2}\right). \tag{45}$$

which gives an E1 gain parameter of

$$C_{\mathrm{E1}}^3 = -\left(\frac{l\omega^2}{4u_0}\right) p_1^2 \left(\frac{J(0,t)}{M u_0^2/2}\right) = \left(\frac{Z_0\gamma_g\gamma_b}{4}\right) p_1^2 \left(\frac{J(0,t)}{M u_0^2/2}\right) = -\left(\frac{Z_0\omega^2}{4u_0u_g}\right) p_1^2 \left(\frac{J(0,t)}{M u_0^2/2}\right) \leq 0. \tag{46}$$

As before, this parameter recovers the classic TWTA parameter with a sign change,

$$C_{\mathrm{E1}} = -C_{\mathrm{E0}}\left(Q \longrightarrow Q_{\mathrm{E1}} = |p_1|\sqrt{|\gamma_g\gamma_b|} \approx |p_1\omega/u_0|\right), \tag{47}$$

and a substitution of an artificial electric-dipole charge as shown.

For a quick order-of-magnitude estimate of the new parameter C_{E1}, let's consider a supersonic beam of fluoromethane (CH_3F, or Freon 41) following Ref. 8. Rough values are $p_1 \approx 0.6$ D (1 debye $\approx 0.021|Q_e|$ nm), $M \approx 44$ amu, $u_0 \approx 165$ m/s, $J(0,t) \approx 10^9$/s, $\omega \approx 2\pi \times 100$ MHz, and $Z_0 \approx 150$ Ohms. Together this gives $C_{\mathrm{E1}}^3 \approx -2.2 \times 10^{-15}$ and thus $C_{\mathrm{E1}} \approx -1.3 \times 10^{-5}$, which has a magnitude not too far from the classic TWTA case. In terms of effective classic TWTA parameters, $Q_{\{E1\}}/|Q_e| \approx 47$ ppm, which in turn gives an effective electron-beam voltage and current of 0.13 μV and 0.0076 pA. Note that these values assumes the synchronous condition ($\gamma_b \approx \gamma_g$).

This case again has the same form of solutions as the classic TWTA, and the sign change in the Pierce parameter switches which mode is amplifying vs decaying. In the end, signal amplification has the same form: $G_{\mathrm{E1}}(x) \approx e^{|C_{\mathrm{E1}}\omega x\sqrt{3}/(2u_0)|}$. For a rough example, using $C_{\mathrm{E1}} \approx -13$ ppm, $u_0 \approx 165$ m/s (or about 0.6 ppm of the speed of light), $\omega \approx 2\pi \times 100$ MHz, and an interaction length of about 10 cm gives $G_{\mathrm{E1}}(10\text{ cm}) \approx 72$. It's surprising that this predicts such a large gain, even with ω reduced significantly in the above calculations. I suspect such gain is unfeasible because of the assumption of aligned moments, and worry that I may have goofed something in calculating it.

V. DISCUSSION

I started this Note originally to learn how TWTAs work, and later expanded it to look for connections with atomic physics and with magnetic braking (see Appendix A). I did this because, while the components of a TWTA seem straightforward, how they work together to amplify wasn't obvious (at least not to me). The approach used here for the classic TWTA is very close to the original Pierce treatment, and as expected, it does expose a mechanism behind this amplification in the small-signal regime. However, it doesn't fully explain it. More in-depth treatments show that TWTAs are actually beam decelerators at heart, in that they amplify by converting some of the kinetic energy of their electron beam into signal energy, thus slowing that electron beam. It'd be interesting to see if the approach used here could be slightly extended to track this energy flow, to perhaps explore saturation, and to show how the ponderomotive force contributes.

Beam decelerators are important to atomic physics because they're commonly used in experiments and devices to slow beams made of other particles like ions,[b] paramagnetic atoms, and polar molecules. These particles primarily have E0, M1, and E1 interactions, respectively. I was curious to see if TWTA-style coupling and amplification was possible for the dipolar interactions (M1 & E1), which are outside the classic E0 case, and the crude treatments given above suggest the answer's a yes. Additionally, helical and cavity TWTAs resemble the Zeeman slowers and Stark decelerators commonly used in these applications, so there's an intriguing similarity of their implementations. Therefore, there's an interesting connection with atomic physics.

One obstacle, however, is that the typical beams of interest to atomic physics are so drastically slow that there aren't any obvious candidates for waveguides that could be synchronous with them. While there are techniques in atomic physics to make sufficiently slow, synchronous traveling waves (e.g., traveling-wave Stark and Zeeman decelerators), they seem to only create moving potential wells, instead of guiding coherent signals. That is, they only use the forward coupling, and more-or-less abandon the backward coupling. Perhaps there's a way to modify such techniques appropriately? Or, maybe there could be a path forward with metamaterials, extreme delay/serpentine guides, or "slow/stopped light" techniques?

In the meantime, the non-synchronous, slow-beam regime is still worth exploring. Appendix A summarizes solutions for this regime, and shows that amplification still occurs for some cases. This regime also seems to have a connection with magnetic braking that is discussed there. Perhaps there could also be interesting opportunities with "stationary beams" (e.g., trapped samples or vapor cells) and their excitations?

In all, these connections seem worth exploring further, in case there might be some beneficial cross pollination. I can imagine some very tentative directions to pursue. For example, perhaps this could lead to novel decelerators, control techniques via the forward coupling, probe techniques via the backward coupling, or even specialty amplifiers, say for low-power applications or for the inverse case of amplifying beam properties. It'd be interesting to see how things change after including the quantum mechanical properties of the particles, whether there are opportunities (or concerns) with induced moments, and to explore backwards-wave-oscillator style variations (e.g., bifilar helix). I'm particularly curious about whether there are opportunities with atomic clocks, for example, with alkali-metal atoms that have ground-state M1 hyperfine transitions. Could you use a guide to make some sort of traveling-wave maser clock? And do non-resonant TWTA-style guides offer a practical way to avoid the line pulling of typical resonant cavities in clocks? Last, but not least, it'd be interesting to explore a connection with free-electron lasers.

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[b] We didn't explicitly treat ions before, but they follow the E0 case. Here's a rough estimate: $M \approx 100$ amu, $u_0 \approx 245$ m/s, and $J(0,t) \approx 10^{10}$/s, gives $C_{\rm ion} \approx 2 \times 10^{-7}$, not too far from the previous M1 case.

Appendix A: Slow-beam solutions and magnetic braking

Consider a particle beam slow enough that $u_b/u_0 = \gamma_g/\gamma_b = \epsilon \ll 1$. For example, a $u_b \approx 100$ m/s beam vs a "slow" $u_g \approx v_c/3$ guide gives $\epsilon \approx 10^{-6}$. The above derivations still apply in this slow-beam regime, but we need to find non-synchronous solutions to the dispersion relations. Proceeding similarly as before, the leading-order approximate solutions for the E0 and M1 cases, which follow the dispersion relation (25), are $\gamma \in \{\gamma_b(1 + i\sqrt{2\epsilon}\, C^{3/2}), \gamma_b(1 - i\sqrt{2\epsilon}\, C^{3/2}), \gamma_g(1-\epsilon C^3), \gamma_g(-1+\epsilon C^3)\}$. Similarly, for the E1 case, which follows the dispersion relation (44), the leading-order approximate solutions are $\gamma \in \{\gamma_b(1 + i\sqrt{2}\, C^{3/2}), \gamma_b(1 - i\sqrt{2}\, C^{3/2}), \gamma_g(1 - \epsilon^2 C^3), \gamma_g(-1 + \epsilon^2 C^3)\}$. In both cases, there's a beam mode with non-negligible gain (and thus beam deceleration) if $C > 0$, but not if $C < 0$.

Naively, I would expect beam deceleration for the M1 case in this slow-beam regime, because of its strong resemblance with a magnetic brake.[c] In particular, notice the similarity with a common classroom demonstration of deceleration of a magnet falling within a metal pipe (e.g., Ref. 9). They're similar in that the falling magnet can be treated as a falling particle with an aligned dipole, and the forward and backward couplings are similar (i.e., the falling particle induces circumferential eddy currents in the pipe). However, they're rather different in that we require an oscillatory current in the guide, and when non-oscillatory, our circumferential currents don't actually complete a circuit, no matter how large the winding density $|n_h|$. Also, we treat an initially uniform beam, not a particle, so the backward coupling is a bit different (i.e., the particle case induces EMF with a dispersive shape about the particle location that washes out for a beam, unless the beam becomes nonuniform, say, via so-called velocity modulation/bunching into packets). Note that shorting both ends of a finite solenoid together wouldn't help, except as the magnet enters and exits the solenoid, because of the symmetry of the EMF shape for constant velocity, which sums to zero.

Nevertheless, it's fun to compute the expected braking of an individual particle for the M1 case. To do this, let's use Ref. 9. The axial magnetic field at a distance z from a current loop is $B_z \approx \mu_0 r^2_{\{\mu_1\}} I_{\{\mu_1\}}/(2z^3) = \mu_0\mu_1/(2\pi z^3)$. Comparing this with Eq. (1) in Ref. 9 gives that the moment $\mu_1 = B_z(2\pi z^3)/\mu_0 = \pi M_r d^3/6$ in terms of their magnetized sphere model with diameter d and remanent magnetization M_r. Using this, the magnetic-drag coefficient of their Eq. (14) becomes $c_m = (45\pi^2/64)(\mu_0\mu_1)^2 \Omega W/D^4$, for a pipe with conductivity Ω, wall thickness W, and diameter D. Using their Eq. (13), the initial braking deceleration is then $a = F_z/M \approx c_m u_0/M$. Numerically, for a copper pipe with $\sigma \approx 4.8 \times 10^7/(\Omega\ \text{m})$ and $D \approx W \approx 1$ cm, and for a particle with $\mu_1 \approx \mu_b$, $M \approx 133$ amu, and $u_0 \approx 245$ m/s, this gives $a \approx 5 \times 10^{-31}$ m/s^2. That's truly tiny! (If this had actually been significant, someone probably would've noticed it, since it'd impact Zeeman slowers.) To get an appreciable effect, say on the same order as typical gravity, you'd have to shrink the pipe diameter down to about 0.1 nm, or atom size. This seems to make sense, since $c_m \propto (d/D)^6$ for the magnetized-sphere model, so $D \approx d$ maximizes the braking force.

Potential future work could explore the connection between models of magnetic braking of a falling magnet within a metal pipe and of the M1 TWTA case. Perhaps there's a way to

[c] Magnetic brakes are also known as eddy current, induction, Faraday, or electric brakes. For more see: `https://en.wikipedia.org/wiki/Eddy_current_brake`

induce non-negligible braking of a falling magnet within a solenoid by exciting a decelerating mode with a signal generator? That could make an interesting experiment or demo, since the signal generator could provide a level of parametric control absent in typical classroom magnetic-braking experiments.

REFERENCES

[1]D. F. G. Minenna, F. André, Y. Elskens, J.-F. Auboin, F. Doveil, J. Puech, and É. Durverdier, "The traveling-wave tube in the history of telecommunication," Eur. Phys. J. H **44**, 1–36 (2019). Appendix A provides an intuitive pedagogical introduction.
Available online: `https://arxiv.org/abs/1803.11497`

[2]P. Wong, P. Zhang, and J. Luginsland, "Recent theory of traveling-wave tubes: a tutorial-review,"' Plasma Res. Express **2**, 023001 (2020).
Available online: `https://www.egr.msu.edu/~pz/tutorial-TWT.pdf`

[3]J. R. Pierce, *Traveling-Wave Tubes,* D. Van Nostrand (1950).
Partially available online: `https://archive.org/details/travelingwavetub0000pier`

[4]A. S. Gilmour, *Klystrons, Traveling Wave Tubes, Magnetrons, Crossed-Field Amplifiers, and Gyrotrons,* Artech House (2011).

[5]L. J. Jasper, "Magnetic bubble traveling-wave amplifier," DTIC ADA065512 (1979).
Available online: `https://apps.dtic.mil/sti/pdfs/ADA065512.pdf`

[6]There are many options on how to model the beam. These functions of position follow an "Eulerian" approach, which requires care with total derivatives and Taylor series to track individual particles. Alternatively, we could model each particle directly in a Lagrangian approach. For more on this, I recommend: A. J. Roberts, *A One-Dimensional Introduction to Continuum Mechanics,* World Scientific (1994).

[7]T. A. Boyer, "The force on a magnetic dipole," Am. J. Phys. **56**, 688–692 (1988).

[8]X. Wu, T. Gantner, M. Koller, M. Zeppenfeld, S. Chervenkov, and G. Rempe. "A cryofuge for cold-collision experiments with slow polar molecules," Science **358**, 645–648 (2017).
Available online: `https://arxiv.org/abs/1710.05988`

[9]S. Land, P. McGuire, N. Bumb, B. P. Mann, and B. B. Yellen, "Electromagnetic braking revisited with a magnetic point dipole model," Am. J. Phys. **84**, 257–262 (2016).