Vector and matrix notation for multiple coordinate systems

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https://bartmcguyer.com/notes/note-20-MatrixNotation.pdf

(Dated: November 20, 2025.)

TL;DR: Handy notation that helps keep track of multiple coordinate systems when working with vectors and matrices in three dimensions, and some references.

This Note shares a nonstandard notation that I found helpful while analyzing the angular errors of mechanical and optical systems in 3D for aerospace applications. There's not much to it... The notation uses square brackets, subscripts, and arrows to explicitly keep track of coordinate systems (or frames of reference) and the transformations between them, which helps minimize silly errors when using multiple frames.

CONTENTS

I. NOTATION

A. Getting started

Let's assume all coordinate systems are right handed. Let's use bold to indicate both vectors and matrices, and reserve a font (here, LaTeX's mathcal) to label coordinate systems. For vectors, let's use capital letters in general, but reserve lower-case letters for unit vectors and hats for basis vectors. For matrices, let's also use capital letters in general, but reserve $\mathbf{R}$ for rotations and other transformations.

To proceed, consider a 3D vector $\mathbf{V}$, a 3×3 matrix $\mathbf{M}$, a Cartesian coordinate system $\mathcal{P}$ with basis unit vectors $\{\hat{\mathbf{x}}_\mathcal{P}, \hat{\mathbf{y}}_\mathcal{P}, \hat{\mathbf{z}}_\mathcal{P}\}$, and another system $\mathcal{Q}$ with basis vectors $\{\hat{\mathbf{x}}_\mathcal{Q}, \hat{\mathbf{y}}_\mathcal{Q}, \hat{\mathbf{z}}_\mathcal{Q}\}$. Let $\mathbf{I}$ denote the 3×3 identity matrix.

B. Vector components

For any vector $\mathbf{V}$, let $[\mathbf{V}]_\mathcal{P}$ denote a column array of its components in the system $\mathcal{P}$,

$$[\mathbf{V}]_\mathcal{P} = (\mathbf{V}\cdot\hat{\mathbf{x}}_\mathcal{P}, \mathbf{V}\cdot\hat{\mathbf{y}}_\mathcal{P}, \mathbf{V}\cdot\hat{\mathbf{z}}_\mathcal{P})^T = \begin{pmatrix}\mathbf{V}\cdot\hat{\mathbf{x}}_\mathcal{P}\\ \mathbf{V}\cdot\hat{\mathbf{y}}_\mathcal{P}\\ \mathbf{V}\cdot\hat{\mathbf{z}}_\mathcal{P}\end{pmatrix}, \tag{1}$$

where T denotes transpose. Using this, $[\mathbf{V}]_\mathcal{Q}$ then denotes a column vector of the components of $\mathbf{V}$ in $\mathcal{Q}$. An optional shorthand for the entries of these columns is $V_n^{\{\mathcal{P}\}}$, where $n \in \{x, y, z\}$. To use numerical values for indices like n, let $\{1, 2, 3\} \equiv \{x, y, z\}$.

C. Matrix components

Similarly, for any matrix $\mathbf{M}$, let $[\mathbf{M}]_\mathcal{P}$ denote a 3×3 array of the components of $\mathbf{M}$ in $\mathcal{P}$,

$$[\mathbf{M}]_\mathcal{P} = \begin{pmatrix} M_{xx}^{\{\mathcal{P}\}} & M_{xy}^{\{\mathcal{P}\}} & M_{xz}^{\{\mathcal{P}\}} \\ M_{yx}^{\{\mathcal{P}\}} & M_{yy}^{\{\mathcal{P}\}} & M_{yz}^{\{\mathcal{P}\}} \\ M_{zx}^{\{\mathcal{P}\}} & M_{zy}^{\{\mathcal{P}\}} & M_{zz}^{\{\mathcal{P}\}} \end{pmatrix}, \tag{2}$$

where the shorthand $M_{xy}^{\{\mathcal{P}\}} = \hat{\mathbf{x}}_\mathcal{P}^T\,\mathbf{M}\,\hat{\mathbf{y}}_\mathcal{P}$.

D. Equation components

Together, the above notation can convert back and forth between an abstract matrix equation and a specific numerical equation by applying brackets as follows:

$$\begin{aligned} \mathbf{U} = \mathbf{M}\,\mathbf{V} &\longleftrightarrow [\mathbf{U} = \mathbf{M}\,\mathbf{V}]_\mathcal{P} \\ &\longleftrightarrow [\mathbf{U}]_\mathcal{P} = [\mathbf{M}\,\mathbf{V}]_\mathcal{P} = [\mathbf{M}]_\mathcal{P}\,[\mathbf{V}]_\mathcal{P} \\ &\longleftrightarrow \begin{pmatrix} U_x^{\{\mathcal{P}\}} \\ U_y^{\{\mathcal{P}\}} \\ U_z^{\{\mathcal{P}\}} \end{pmatrix} = \begin{pmatrix} M_{xx}^{\{\mathcal{P}\}} & M_{xy}^{\{\mathcal{P}\}} & M_{xz}^{\{\mathcal{P}\}} \\ M_{yx}^{\{\mathcal{P}\}} & M_{yy}^{\{\mathcal{P}\}} & M_{yz}^{\{\mathcal{P}\}} \\ M_{zx}^{\{\mathcal{P}\}} & M_{zy}^{\{\mathcal{P}\}} & M_{zz}^{\{\mathcal{P}\}} \end{pmatrix} \begin{pmatrix} V_x^{\{\mathcal{P}\}} \\ V_y^{\{\mathcal{P}\}} \\ V_z^{\{\mathcal{P}\}} \end{pmatrix}, \end{aligned} \tag{3}$$

where $\mathbf{U}$ is another vector. Note that all components here share the same frame $\mathcal{P}$.

E. Change of basis for vectors and matrices

Let $\mathbf{R}_{\mathcal{Q}\leftarrow\mathcal{P}}$ denote a 3×3 array that converts the components of a vector $\mathbf{V}$ in $\mathcal{P}$ to its components in $\mathcal{Q}$, via

$$[\mathbf{V}]_\mathcal{Q} = \mathbf{R}_{\mathcal{Q}\leftarrow\mathcal{P}}\,[\mathbf{V}]_\mathcal{P}. \tag{4}$$

This array describes a "passive" (or change-of-basis) transformation that keeps vectors fixed but changes coordinate systems, and is given by dot products (or directional cosines) between all of the basis vectors of both systems as follows:

$$\mathbf{R}_{\mathcal{Q}\leftarrow\mathcal{P}} = \begin{pmatrix} [\hat{\mathbf{x}}_{\mathcal{Q}}]_{\mathcal{P}}^T \\ [\hat{\mathbf{y}}_{\mathcal{Q}}]_{\mathcal{P}}^T \\ [\hat{\mathbf{z}}_{\mathcal{Q}}]_{\mathcal{P}}^T \end{pmatrix} = \big([\hat{\mathbf{x}}_{\mathcal{P}}]_{\mathcal{Q}},\ [\hat{\mathbf{y}}_{\mathcal{P}}]_{\mathcal{Q}},\ [\hat{\mathbf{z}}_{\mathcal{P}}]_{\mathcal{Q}}\big) = \begin{pmatrix} \hat{\mathbf{x}}_{\mathcal{Q}}\cdot\hat{\mathbf{x}}_{\mathcal{P}}, & \hat{\mathbf{x}}_{\mathcal{Q}}\cdot\hat{\mathbf{y}}_{\mathcal{P}}, & \hat{\mathbf{x}}_{\mathcal{Q}}\cdot\hat{\mathbf{z}}_{\mathcal{P}} \\ \hat{\mathbf{y}}_{\mathcal{Q}}\cdot\hat{\mathbf{x}}_{\mathcal{P}}, & \hat{\mathbf{y}}_{\mathcal{Q}}\cdot\hat{\mathbf{y}}_{\mathcal{P}}, & \hat{\mathbf{y}}_{\mathcal{Q}}\cdot\hat{\mathbf{z}}_{\mathcal{P}} \\ \hat{\mathbf{z}}_{\mathcal{Q}}\cdot\hat{\mathbf{x}}_{\mathcal{P}}, & \hat{\mathbf{z}}_{\mathcal{Q}}\cdot\hat{\mathbf{x}}_{\mathcal{P}}, & \hat{\mathbf{z}}_{\mathcal{Q}}\cdot\hat{\mathbf{z}}_{\mathcal{P}} \end{pmatrix}. \tag{5}$$

Successive change-of-basis arrays combine as

$$\mathbf{R}_{\mathcal{P}\leftarrow\mathcal{Q}}\mathbf{R}_{\mathcal{Q}\leftarrow\mathcal{R}} = \mathbf{R}_{\mathcal{P}\leftarrow\mathcal{R}} \tag{6}$$

when neighboring frames match (in above, $\mathcal{Q}$), where $\mathcal{R}$ is another frame. In this arrow notation, an inverse (or transpose) is equivalent to swapping frame symbols:

$$\mathbf{R}_{\mathcal{Q}\leftarrow\mathcal{P}}^{-1} = \mathbf{R}_{\mathcal{Q}\leftarrow\mathcal{P}}^{T} = \mathbf{R}_{\mathcal{P}\leftarrow\mathcal{Q}}. \tag{7}$$

Using this, the conversion of the components of a matrix $\mathbf{M}$ in $\mathcal{P}$ to those in $\mathcal{Q}$ is given by the similarity transformation

$$[\mathbf{M}]_{\mathcal{Q}} = \mathbf{R}_{\mathcal{Q}\leftarrow\mathcal{P}}\,[\mathbf{M}]_{\mathcal{P}}\,\mathbf{R}_{\mathcal{P}\leftarrow\mathcal{Q}}. \tag{8}$$

Note that this notation lacks the brackets introduced above to give components in a frame. This visual styling is for two reasons: (1^{st}) I found that this helps remind that these arrays are different from other matrices and don't transform following (8). (2^{nd}) In practice, these arrays are often imperfect estimates with important errors, so it can be convenient to reserve brackets to indicate a true (error free) change of basis as follows. Note that we can extend (3) to show that these change-of-basis arrays are mixed-basis components of $\mathbf{I}$:

$$\mathbf{V} = \mathbf{I}\,\mathbf{V} \longleftrightarrow [\mathbf{V}]_{\mathcal{P}} = [\mathbf{I}]_{\mathcal{P}}\,[\mathbf{V}]_{\mathcal{P}} = [\mathbf{I}]_{\mathcal{P}\leftarrow\mathcal{P}}\,[\mathbf{V}]_{\mathcal{P}} = [\mathbf{I}]_{\mathcal{P}\leftarrow\mathcal{Q}}\,[\mathbf{V}]_{\mathcal{Q}} = [\mathbf{I}]_{\mathcal{P}\leftarrow\mathcal{R}}\,[\mathbf{V}]_{\mathcal{R}} = \dots \tag{9}$$

Thus, when $\mathbf{R}_{\mathcal{P}\leftarrow\mathcal{Q}}$ is free of error, this gives $\mathbf{R}_{\mathcal{P}\leftarrow\mathcal{Q}} = [\mathbf{I}]_{\mathcal{P}\leftarrow\mathcal{Q}}$ (or $\mathbf{I}_{\mathcal{P}\leftarrow\mathcal{Q}}$ for short). Otherwise, when instead $\mathbf{R}_{\mathcal{Q}\leftarrow\mathcal{P}} \approx [\mathbf{I}]_{\mathcal{Q}\leftarrow\mathcal{P}}$, it can be convenient to keep these quantities separate by using $[\mathbf{I}]_{\mathcal{P}\leftarrow\mathcal{Q}}$ for the true change of basis and $\mathbf{R}_{\mathcal{Q}\leftarrow\mathcal{P}}$ for an estimate of it. In this case, some care's needed to track which quantities were used with (8) and (6). Finally, if preferable, an optional alternative to arrows is to label left and right systems via ${}_{\mathcal{P}}\mathbf{R}_{\mathcal{Q}}$ and ${}_{\mathcal{P}}[\mathbf{M}]_{\mathcal{Q}}$, such that ${}_{\mathcal{P}}[\mathbf{M}]_{\mathcal{P}} = [\mathbf{M}]_{\mathcal{P}}$.

II. REFERENCE

A. Basic rotations

The following matrices describe "active" rotations that move vectors while keeping the coordinate system fixed, where the shorthand "$c\theta$" $=\cos(\theta)$ and "$s\theta$" $=\sin(\theta)$:

$$\mathbf{R}_{\mathbf{x}}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{pmatrix}, \quad \mathbf{R}_{\mathbf{y}}(\theta) = \begin{pmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{pmatrix}, \quad \text{and} \quad \mathbf{R}_{\mathbf{z}}(\theta) = \begin{pmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

These rotate vectors by the angle θ following the right-hand rule about the axis indicated in their subscript. Note that $\mathbf{R_n}(-\theta) = \mathbf{R_n}(\theta)^T$ and $\mathbf{R_n}(\theta_1)\mathbf{R_n}(\theta_2) = \mathbf{R_n}(\theta_1 + \theta_2)$.

To use one of these basic rotations in an "active" sense, you must pick a coordinate system. For example, setting $[\mathbf{M}]_\mathcal{P} = \mathbf{R_x}(\theta)$ gives a matrix $\mathbf{M}$ that describes a right-hand rotation by θ about $\hat{\mathbf{x}}_\mathcal{P}$. That is, the vector $\mathbf{M}\,\mathbf{V}$ is rotated that way compared to $\mathbf{V}$. Similarly, products of these rotations can construct "passive" rotations, such as $\mathbf{R}_{\mathcal{P}\leftarrow\mathcal{Q}} = \mathbf{R_z}(\theta_3)\mathbf{R_y}(\theta_2)\mathbf{R_x}(\theta_1)$, using Euler or Tait-Bryan angle parametrizations.

B. Axis-angle rotation

Let $\mathbf{R_a}(\omega, [\mathbf{n}])$ denote the axis-angle rotation matrix

$$\mathbf{R_a}(\omega, [\mathbf{n}]) = \cos(\omega)\mathbf{I} + [1 - \cos(\omega)] \begin{pmatrix} n_x^2 & n_x n_y & n_x n_z \\ n_x n_y & n_y^2 & n_y n_z \\ n_x n_z & n_y n_z & n_z^2 \end{pmatrix} + \sin(\omega) \begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}, \tag{11}$$

which describes an "active" right-handed rotation by the angle ω about the unit vector $\mathbf{n}$ with explicit components $[\mathbf{n}] \equiv (n_x, n_y, n_z)^T$ in an unspecified frame. Again, to use this, you must pick a coordinate system. For example, setting $[\mathbf{M}]_\mathcal{P} = \mathbf{R_a}(\omega, [\mathbf{m}]_\mathcal{P})$ gives a matrix $\mathbf{M}$ that describes a right-hand rotation by ω about a unit vector $\mathbf{m}$ and sets the components $(n_x, n_y, n_z)^T = [\mathbf{m}]_\mathcal{P}$ in (11).

C. First-order angular error matrix

The following handy matrix introduces small angular offsets

$$[\mathbf{M}_\text{error}]_\mathcal{P} = \begin{pmatrix} 1 & -\delta_z & \delta_y \\ \delta_z & 1 & -\delta_x \\ -\delta_y & \delta_x & 1 \end{pmatrix} \approx \mathbf{R_x}(\delta_x)\,\mathbf{R_y}(\delta_y)\,\mathbf{R_z}(\delta_z) \quad \text{for} \quad |\delta_n| \ll 1. \tag{12}$$

that can simulate attitude or transformation errors, depending on where it's inserted in some analysis. Note that the order of the $\mathbf{R_n}(\theta_n)$ on the right-hand side doesn't matter.

D. Comparing rotations (angular distance and axis of misalignment)

Consider two rotation matrices $\mathbf{R}_1$ and $\mathbf{R}_2$ and a unit vector $\mathbf{x}$. One way to compare these matrices is to compute their so-called angular distance, or the largest possible angle between $\mathbf{R}_1\,\mathbf{x}$ and $\mathbf{R}_2\,\mathbf{x}$ for any $\mathbf{x}$:

$$\psi_\text{max} = \psi_\text{max}(\mathbf{R}_1, \mathbf{R}_2) = \left| \arccos\left\{ \frac{1}{2}\left[\text{Tr}\left(\mathbf{R}_1^T\,\mathbf{R}_2 \right) - 1 \right] \right\} \right| \in [0, \pi], \tag{13}$$

which is zero if $\mathbf{R}_1 = \mathbf{R}_2$. Note that trace is invariant to (8), so it can be calculated in any preferred frame. This angle follows from noting $(\mathbf{R}_1\,\mathbf{x}) \cdot (\mathbf{R}_2\,\mathbf{x}) = \mathbf{x} \cdot \left(\mathbf{R}_1^T\,\mathbf{R}_2\,\mathbf{x}\right)$ and using $\mathbf{R}_{\mathbf{a}}(\psi_{\max}, [\mathbf{n}]_{\mathcal{P}}) = [\mathbf{R}_1^T\,\mathbf{R}_2]_{\mathcal{P}}$, which also gives the corresponding axis for this angle as

$$[\mathbf{n}]_{\mathcal{P}} = \frac{\left(S_{zy}^{\{\mathcal{P}\}} - S_{yz}^{\{\mathcal{P}\}}, S_{xz}^{\{\mathcal{P}\}} - S_{zx}^{\{\mathcal{P}\}}, S_{yx}^{\{\mathcal{P}\}} - S_{xy}^{\{\mathcal{P}\}}\right)}{2\sin(\psi_{\max})}, \tag{14}$$

where $\mathbf{S} = \mathbf{R}_1^T\,\mathbf{R}_2$ and the shorthand $S_{xy}^{\{\mathcal{P}\}} = \hat{\mathbf{x}}_{\mathcal{P}}^T\,\mathbf{S}\,\hat{\mathbf{y}}_{\mathcal{P}}$.